Government of South Australia
SACE Board of SA
External Examination 2010

2010 SPECIALIST MATHEMATICS

FOR OFFICE USE ONLY

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RE-MARKED

ATTACH SACE REGISTRATION NUMBER LABEL TO THIS BOX

Graphics calculator ☐
Brand ____________
Model ________
Computer software ☐

Friday 12 November: 9 a.m.

Time: 3 hours

Pages: 41
Questions: 16

Examination material: one 41-page question booklet
one SACE registration number label

Approved dictionaries, notes, calculators, and computer software may be used.

Instructions to Students

1. You will have 10 minutes to read the paper. You must not write in your question booklet or use a calculator during this reading time but you may make notes on the scribbling paper provided.

2. This paper consists of three sections:

   Section A (Questions 1 to 10) 75 marks
   Answer ***all*** questions in Section A.

   Section B (Questions 11 to 14) 60 marks
   Answer ***all*** questions in Section B.

   Section C (Questions 15 and 16) 15 marks
   Answer ***one*** question from Section C.

3. Write your answers in the spaces provided in this question booklet. There is no need to fill all the space provided. You may write on pages 15, 23, and 31 if you need more space, making sure to label each answer clearly.

4. Appropriate steps of logic and correct answers are required for full marks.

5. Show all working in this booklet. (You are strongly advised ***not*** to use scribbling paper. Work that you consider incorrect should be crossed out with a single line.)

6. Use only black or blue pens for all work other than graphs and diagrams, for which you may use a sharp dark pencil.

7. State all answers correct to three significant figures, unless otherwise stated or as appropriate.

8. Diagrams, where given, are not necessarily drawn to scale.

9. The list of mathematical formulae is on page 41. You may remove the page from this booklet before the examination begins.

10. Complete the box on the top right-hand side of this page with information about the electronic technology you are using in this examination.

11. Attach your SACE registration number label to the box at the top of this page.

SECTION A (Questions 1 to 10)

(75 marks)

*Answer **all** questions in this section.*

QUESTION 1 (6 marks)

The equation of a plane is $2x - y + 4z = 6$ and the equation of a line in space is $x = 2 + t,\ y = 1 - t,\ z = 3 + 2t,$ where t is a parameter.

(a) State a direction vector normal to the plane.

(1 mark)

(b) State a direction vector of the line.

(1 mark)

(c) Hence find the acute angle between the line and the plane.

(4 marks)

QUESTION 2 (5 marks)

Consider the parametric equations of a curve $(x(t),\ y(t))$, where

$$\begin{aligned} x(t) &= 2t^3 + 6t \\ y(t) &= 6\sin t - 3t \end{aligned} \quad \text{for } t \geq 0.$$

(a) Show clearly that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\cos t - \frac{1}{2}}{t^2 + 1}$.

(3 marks)

(b) Find the smallest exact value of t at which the curve has a stationary point.

(2 marks)

QUESTION 3 (6 marks)

In Figure 1 point P moves between points A and B on a semicircular arc constructed on a side of equilateral triangle ABC.

Point M is the midpoint of CB, and P is never at A or B.

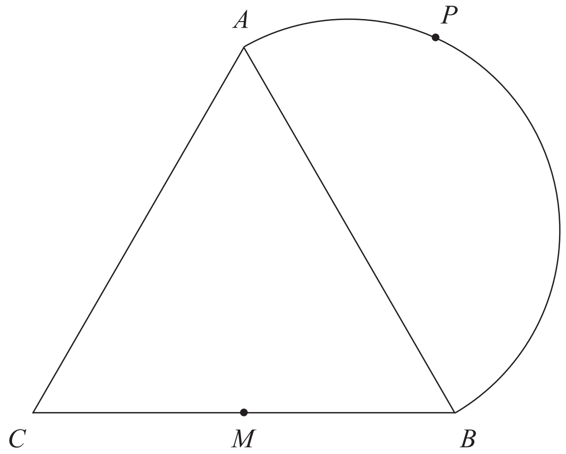

Figure 1

(a) Prove that $APBM$ is a cyclic quadrilateral.

(3 marks)

(b) Hence prove that $\angle AMP = \angle ABP$.

(1 mark)

(c) Prove that $\angle MPB$ has a constant value whatever the position of P on the semicircular arc.

(2 marks)

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QUESTION 4 (8 marks)

Sociologists can study the spread of a new fashion by modelling the rate at which the fashion spreads. For one such model the rate of spread is given by the differential equation

$$\frac{dN}{dt} = \frac{0.5N(10000 - N)}{10000}$$

where N is the number of people who follow the fashion and t is in weeks.

Initially there are 100 people following the fashion.

(a) Show that $\dfrac{10000}{N(10000-N)} = \dfrac{1}{N} + \dfrac{1}{10000-N}$.

(1 mark)

(b) Solve the differential equation given above with the initial condition $t = 0$, $N = 100$ to show that

$$N = \frac{10000}{1 + 99e^{-0.5t}}.$$

(5 marks)

(c) What is the estimated time at which the fashion is spreading at the greatest rate?

(2 marks)

PLEASE TURN OVER

QUESTION 5 (7 marks)

(a) Find α and β if $x^2+1=(x+\alpha)(x+\beta)$.

(1 mark)

(b) Use the result of part (a) to show that x^2+1 is a factor of $x^{2010}+1$.

(2 marks)

(c) (i) Given that $x^{2011}+1=x\left(x^{2010}+1\right)+T(x)$, find $T(x)$.

(2 marks)

(ii) Hence find the remainder when $x^{2011}+1$ is divided by $x^{2}+1$.

(2 marks)

QUESTION 6 (9 marks)

It is known that $\sin 2\theta = 2\sin\theta\cos\theta$.

(a) Show that:

(i) $\sin 4\theta = 4\sin\theta\cos\theta\cos 2\theta$.

(1 mark)

(ii) $\sin 8\theta = 8\sin\theta\cos\theta\cos 2\theta\cos 4\theta$.

(2 marks)

(b) Use an inductive argument to show that $\sin\left(2^n\theta\right) = 2^n \sin\theta\cos\theta\cos 2\theta \ldots \cos\left(2^{n-1}\theta\right)$ for all positive integers n.

(4 marks)

(c) Hence find $\int\left(\sin\theta\cos\theta\cos 2\theta\cos 4\theta\cos 8\theta\cos 16\theta\cos 32\theta\right)\mathrm{d}\theta$.

(2 marks)

PLEASE TURN OVER

QUESTION 7 (9 marks)

A function $y = f(x)$ has derivative $f'(x) = \sin\sqrt{x}$ and $y = 0.5$ when $x = 0$. Figure 2 shows the slope field for this differential equation.

(a) Draw the solution curve on Figure 2.

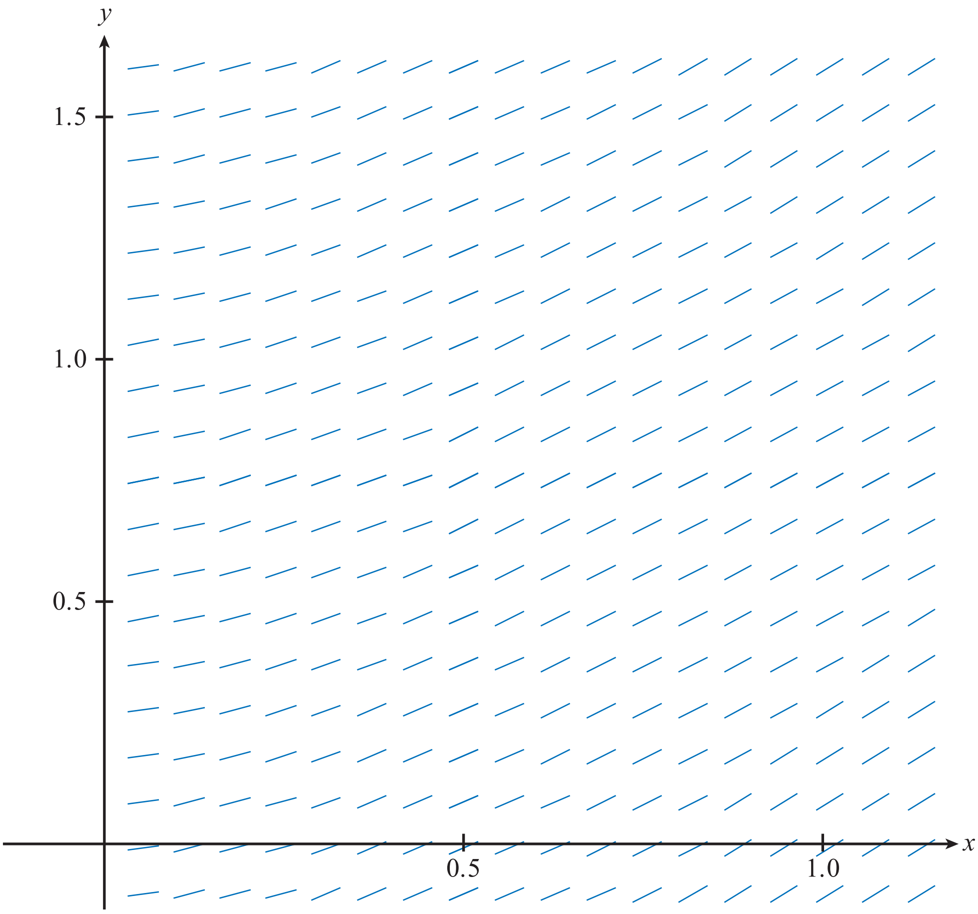


Figure 2

(3 marks)

(b) The equations for Euler's method are

$$x_{n+1} = x_n + h$$
$$y_{n+1} = y_n + hf'(x_n).$$

Use ten steps of Euler's method to calculate an estimate for $y(1)$. You may use all rows of the table if it helps you, but you do not need to complete the shaded rows.

n	h	x_n	y_n	$x_{n+1} = x_n + h$	$y_{n+1} = y_n + hf'(x_n)$
0		0	0.5		

(3 marks)

(c) Considering the shape of the solution curve in part (a), state whether your estimate for $y(1)$ is an overestimate or an underestimate. Justify your answer.

(2 marks)

(d) Explain how you could obtain a better estimate for $y(1)$.

(1 mark)

QUESTION 8 (6 marks)

Figure 3 shows equilateral triangle *ABC*.

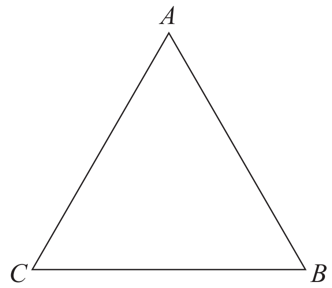


Figure 3

(a) Let $\overrightarrow{CA} = \boldsymbol{a}$, $\overrightarrow{CB} = \boldsymbol{b}$.

(i) On Figure 3, draw the vector $\overrightarrow{BD} = -2\boldsymbol{a}$. (1 mark)

(ii) Prove that $\boldsymbol{b} \bullet (\boldsymbol{b} - 2\boldsymbol{a}) = 0$.

(3 marks)

(b) On Figure 3, illustrate the result from part (a)(ii). (2 marks)

You may write on this page if you need more space to finish your answers. Make sure to label each answer carefully (e.g. 'Question 8(a)(ii) continued').

QUESTION 9 (8 marks)

In Figure 4 $\angle ACB$ is initially a right angle.

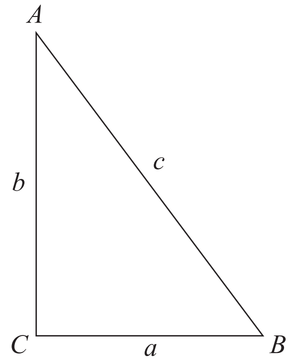


Figure 4

(a) (i) Which one of the following statements is true?

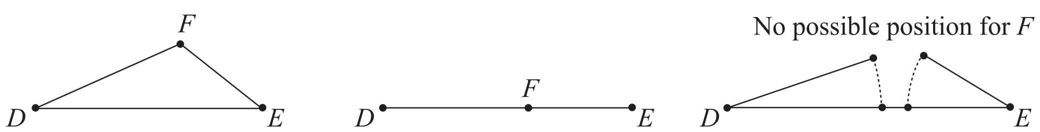
$$a+b<c..................(1)$$
$$a+b=c..................(2)$$
$$a+b>c..................(3)$$

(1 mark)

(ii) Justify your answer to part (a)(i).

(1 mark)

(b) The following diagrams represent attempts to draw a triangle *DEF*, where $DE=c^2$, $EF=a^2$, and $FD=b^2$, and c, a, and b are the side lengths of a triangle *ABC*.



(i) Copy the diagram that you think correctly illustrates the result of an attempt to draw triangle *DEF*, given that triangle *ABC* is right-angled as in Figure 4 above.

(1 mark)

(ii) Justify your answer to part (b)(i).

(1 mark)

(c) Consider a change to triangle ABC in which $\angle ACB$ becomes acute and AB remains the longest side in the triangle.

(i) From part (b), copy the diagram that you think correctly illustrates the result of an attempt to draw triangle DEF, given that triangle ABC is now an acute-angled triangle.

(1 mark)

(ii) Justify your answer to part (c)(i).

(1 mark)

(d) Consider another change to triangle ABC in which $\angle ACB$ becomes obtuse.

(i) From part (b), copy the diagram that you think correctly illustrates the result of an attempt to draw triangle DEF, given that triangle ABC is now an obtuse-angled triangle.

(1 mark)

(ii) Justify your answer to part (d)(i).

(1 mark)

QUESTION 10 (11 marks)

(a) State the centre and the radius of $|z+1| = \frac{1}{4}$.

(2 marks)

(b) Graph the circle on the Argand diagram in Figure 5.

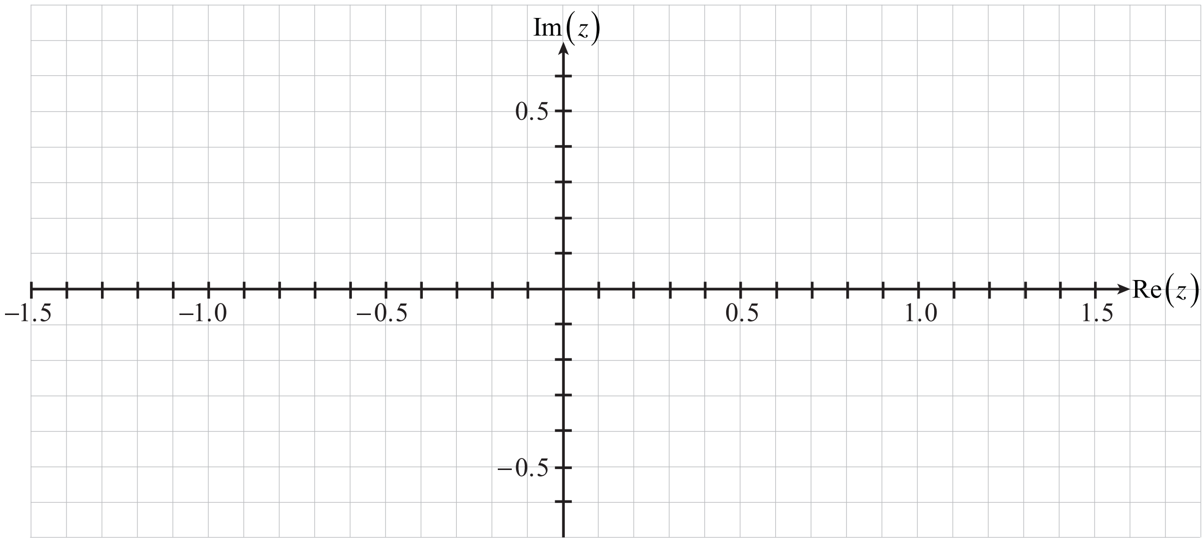


Figure 5 (1 mark)

(c) For the quadratic iteration $z \to z^2 + c$, $z_0 = -0.7 - 0.1i$, $c = -0.78 - 0.04i$:

(i) find z_1, z_2, and z_3.

(2 marks)

(ii) plot and label the points corresponding to c, z_1, z_2, and z_3 on the Argand diagram in Figure 5. (2 marks)

(d) Consider the long-term behaviour of the quadratic iteration
$z \to z^2 + c,\ z_0 = 0,\ c = -0.78 - 0.04i$.

(i) Investigate this long-term behaviour and use the table to record your findings.

n	z_n
0	0
1	
2	

(2 marks)

(ii) For your last entry in the table calculate $|z_n|$.

(1 mark)

(iii) Describe the apparent long-term behaviour of this iteration.

(1 mark)

SECTION B (Questions 11 to 14)

(60 marks)

*Answer **all** questions in this section.*

QUESTION 11 (16 marks)

Figure 6 shows the parallelepiped $ABCDEFGH$ with vertices $A(1, -1, 0)$, $B(-3, 2, 1)$, $D(2, 1, 2)$, and $F(3, 2, -2)$.

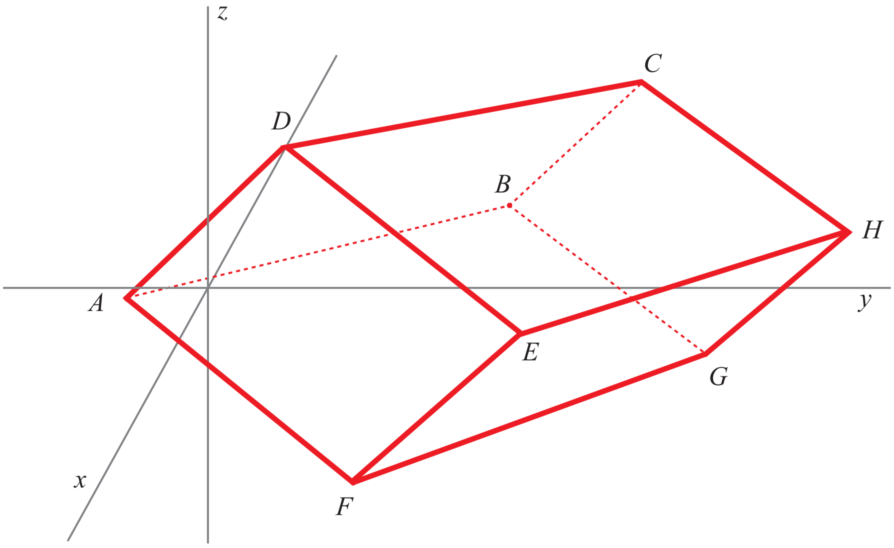


Figure 6

(a) (i) Find $\overrightarrow{AD}$, $\overrightarrow{AB}$, and $\overrightarrow{AF}$.

(2 marks)

(ii) Find the coordinates of point C.

(1 mark)

(iii) Calculate the volume of the parallelepiped.

(2 marks)

(b) (i) Find $\overrightarrow{AF} \times \overrightarrow{AD}$.

(2 marks)

(ii) Show that the equation of the plane *ADEF* is $10x - 6y + z = 16$.

(1 mark)

(iii) Find the equation of the plane *BCHG*.

(1 mark)

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(c) Justify that the diagonal AH meets the diagonal CF at $P\left(\frac{1}{2}, 3, \frac{1}{2}\right)$.

(2 marks)

(d) Find the distance between point P and the plane $ADEF$.

(2 marks)

(e) Find, correct to three significant figures, the coordinates of two points on the surface of the parallelepiped that are the distance from P as found in part (d).

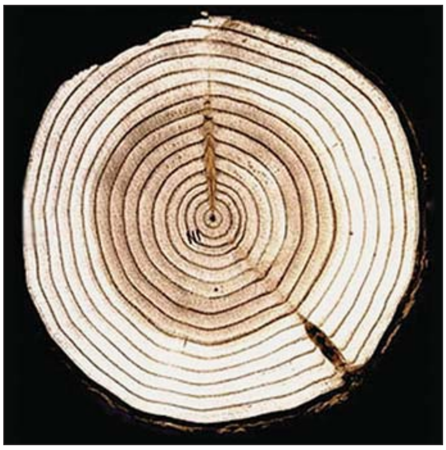
(3 marks)

You may write on this page if you need more space to finish your answers. Make sure to label each answer carefully (e.g. 'Question 8(a)(ii) continued').

QUESTION 12 (14 marks)

The shape of tree trunks can be modelled as cylinders of radius r metres and height h metres. Thus the volume of the trunk is $V = \pi r^2 h$.

The growth of a tree trunk can be estimated by viewing its growth rings, as shown in the photograph. The width of a growth ring represents the radial growth in 1 year.



Source: http://cfs.nrcan.gc.ca/news/278

(a) Consider the cross-section of a tree trunk where the outermost growth ring is 0.032 metres wide. In this year the height of the tree was changing at a rate of 0.2 metres per year.

(i) Give values for $\frac{\mathrm{d}r}{\mathrm{d}t}$ and $\frac{\mathrm{d}h}{\mathrm{d}t}$ for this year.

(2 marks)

(ii) Show that the rate of change of a tree trunk's volume is

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \pi r\left(2h\frac{\mathrm{d}r}{\mathrm{d}t} + r\frac{\mathrm{d}h}{\mathrm{d}t}\right).$$

(2 marks)

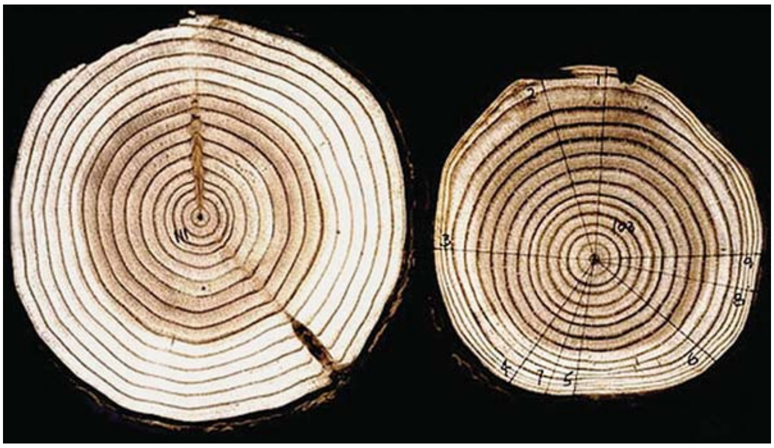
(iii) Use the growth rate values from part (a)(i) to estimate $\frac{dV}{dt} = \pi r \left(2h \frac{dr}{dt} + r \frac{dh}{dt} \right)$, the rate of change of volume of a tree trunk of height 5 metres and volume 1.41 cubic metres.

(2 marks)

(b) The tree shown in the photograph on the right is infected by Armillaria ostoyae root disease, which is caused by a fungus. A major symptom of the disease is a reduction in the tree's growth.

The effect of the disease can be seen in the photograph below, which shows trees of the same age.

Source: D. Morrison, Natural Resources Canada, Canadian Forest Service, http://imfc.cfl.scf.rncan.gc.ca/images-eng.asp?geID=78



Trunk from a healthy tree Trunk from a diseased tree

Source: http://cfs.nrcan.gc.ca/news/278

The effect of the disease within a tree trunk is modelled by the differential equation

$$\frac{dV}{dt} = -0.0639V$$

where V is the volume of the trunk that is not affected by the disease (i.e. the unaffected volume) and t is measured in years.

(i) Given that V_0 is the initial volume of the trunk of a healthy tree, solve the differential equation to show that the unaffected volume is $V = V_0 e^{-0.0639t}$.

(3 marks)

(ii) Find the time taken for 40% of the initial volume to become affected by the disease.

(2 marks)

(c) Suppose that a healthy tree with an initial volume of 1.41 cubic metres becomes infected by the disease.

Use the differential equation in part (b) to show that the disease causes a loss of unaffected volume of 0.0901 cubic metres per year.

(1 mark)

(d) Use your answers to explain the difference between the trunk from a healthy tree and the trunk from a diseased tree as illustrated in part (b).

(2 marks)

QUESTION 13 (16 marks)

(a) Express in exact $a+bi$ form

$$\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}-\frac{i\sqrt{2}}{2}\right).$$

(2 marks)

(b) Write in polar form:

(i) $\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right).$

(1 mark)

(ii) $\left(\frac{\sqrt{2}}{2}-\frac{i\sqrt{2}}{2}\right).$

(1 mark)

(iii) $\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}-\frac{i\sqrt{2}}{2}\right)$.

(1 mark)

(c) Using the results of part (a) and part (b), show exactly that

$$\cos\frac{\pi}{12}=\frac{\sqrt{6}+\sqrt{2}}{4} \text{ and } \sin\frac{\pi}{12}=\frac{\sqrt{6}-\sqrt{2}}{4}.$$

(1 mark)

(d) (i) Using De Moivre's theorem, find the distinct solutions of $z^4=2+2i\sqrt{3}$. Give your solutions in the form $r\,\text{cis}\,\theta$.

(4 marks)

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(ii) Let the solutions of $z^4 = 2 + 2i\sqrt{3}$ be z_1, z_2, z_3, and z_4, with z_1 in the first quadrant and arguments increasing anticlockwise from the positive $\text{Re}(z)$ axis.

Draw and label z_1, z_2, z_3, and z_4 on the Argand diagram in Figure 7.

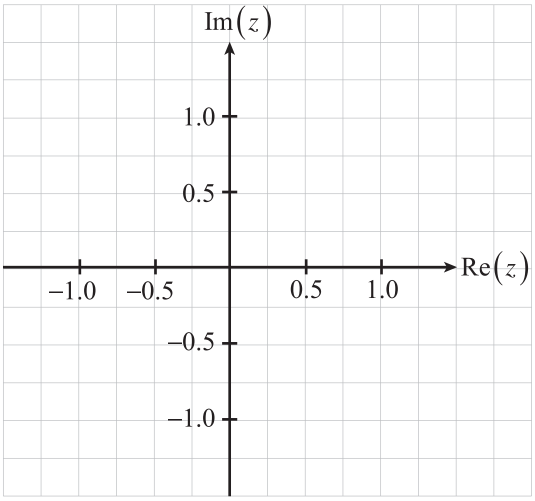

Figure 7

(2 marks)

(e) Using your diagram from part (d)(ii):

(i) show that $|z_1 - z_2| = 2$.

(2 marks)

(ii) find $|z_1 - z_2| + |z_1 - z_4| + |z_1 - z_3|$.

(2 marks)

You may write on this page if you need more space to finish your answers. Make sure to label each answer carefully (e.g. 'Question 8(a)(ii) continued').

PLEASE TURN OVER

QUESTION 14 (14 marks)

Let $f(x) = \dfrac{\cos x}{3 + 2\sin x}$.

(a) (i) On the axes in Figure 8, sketch the graph of $f(x)$ for $0 \le x \le 2\pi$.

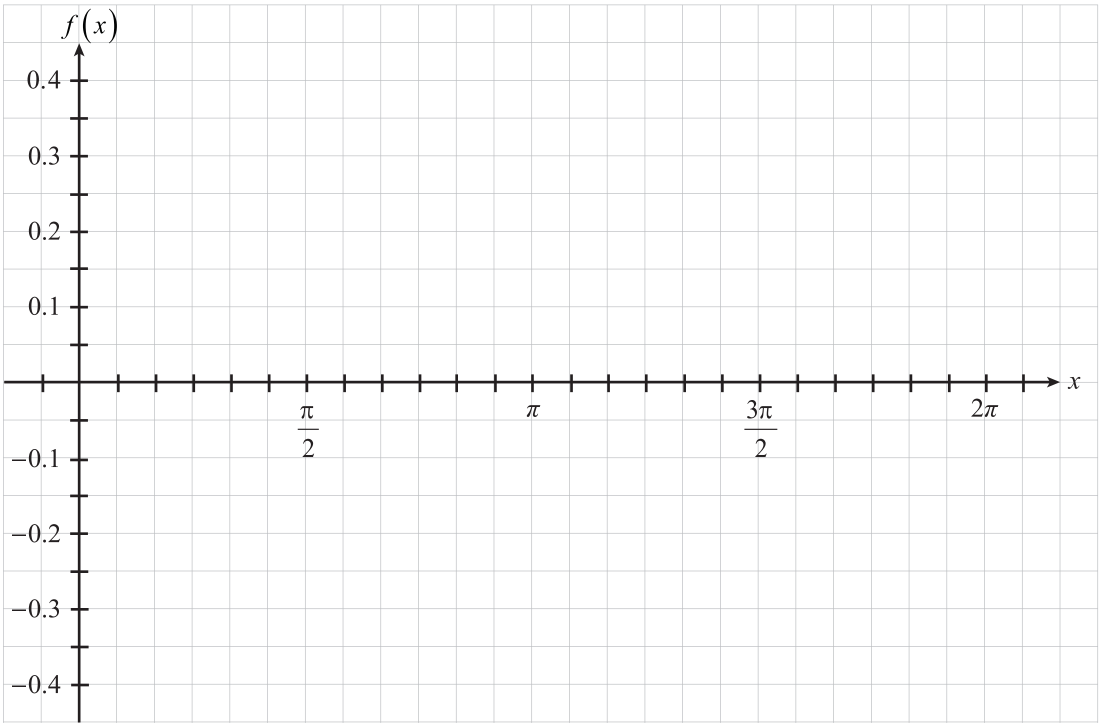

Figure 8 (3 marks)

(ii) Find exact values for the y-intercept and the zeros.

(2 marks)

(b) Find $\int f(x)\,dx$.

(3 marks)

(c) Hence show that $\int_0^A f(x)\,dx = \ln\sqrt{1+\frac{2}{3}\sin A}$, where A is a constant.

(2 marks)

(d) (i) Using your graph from part (a)(i), or otherwise, find the exact value of A where $0 \le A \le 2\pi$, for which $\int_0^A f(x)\,dx$ is a minimum.

(1 mark)

(ii) Hence find the exact value of the minimum.

(1 mark)

(e) (i) Using your graph from part (a)(i), or otherwise, find the exact value of A where $0 \le A \le 2\pi$, for which $\int_0^A f(x)\,dx$ is a maximum.

(1 mark)

(ii) Hence find the exact value of the maximum.

(1 mark)

SECTION C (Questions 15 and 16)

(15 marks)

*Answer **one** question from this section, **either** Question 15 **or** Question 16.*

Answer ***either*** *Question 15* ***or*** *Question 16.*

QUESTION 15 (15 marks)

As the purple sea snail (*Janthina janthina*) grows, the lateral dimensions of its shell can be modelled by the differential system

$$\begin{cases} x' = \dfrac{x}{2} + 3y \\ y' = -3x + \dfrac{y}{2}. \end{cases}$$

Figure 9 illustrates the slope field for this differential system. The point plotted on the x-axis is at $(1, 0)$.

Source: M. Mitchell, www.spacecoastbeachbuzz.com

(a) On Figure 9 draw the solution curve that starts at the point $(1, 0)$ and moves in a clockwise direction.

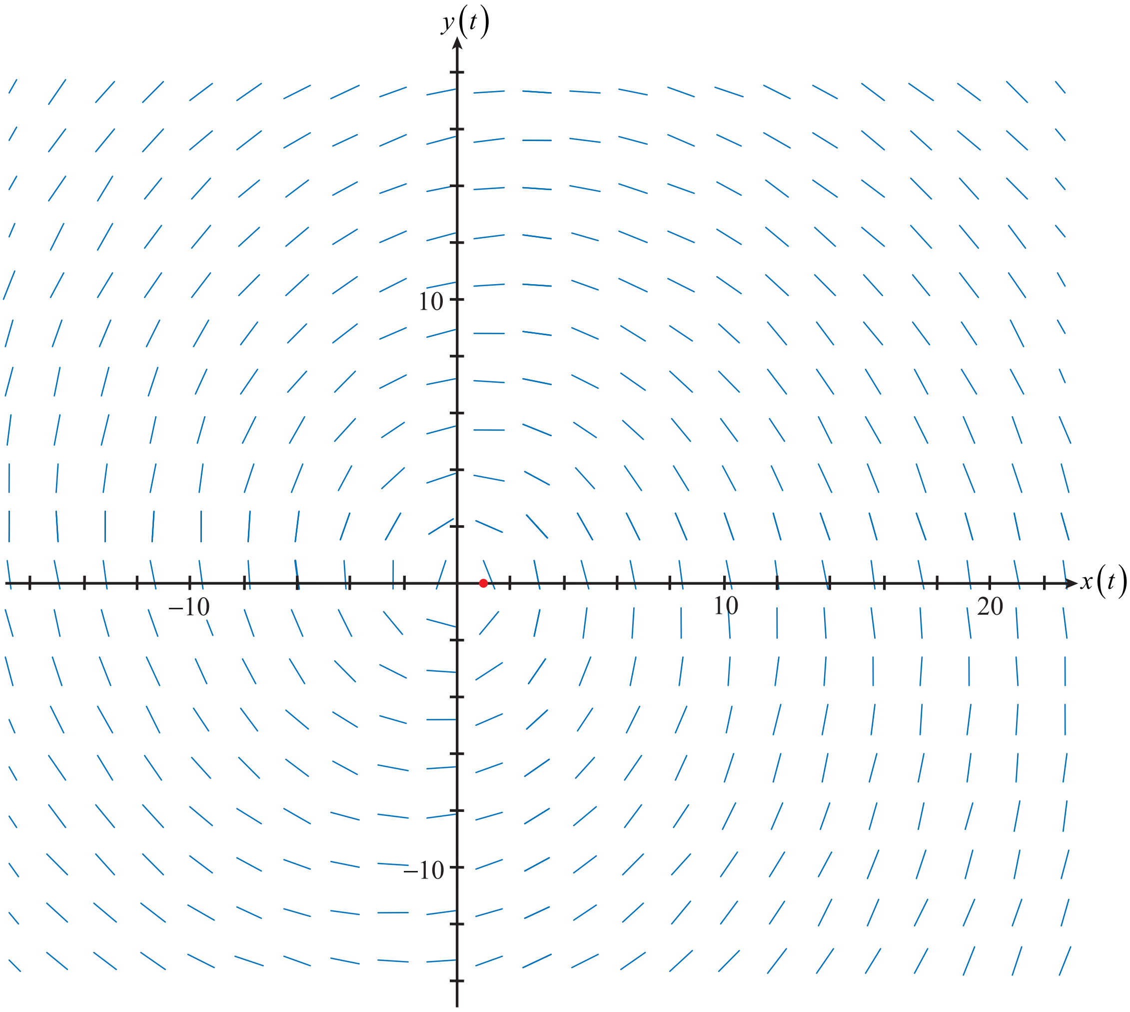

Figure 9 (3 marks)

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(b) To answer this part, use the given differential system (reproduced from page 35):

$$\begin{cases} x' = \dfrac{x}{2} + 3y \\ y' = -3x + \dfrac{y}{2}. \end{cases}$$

This differential system has a solution of the form

$$\begin{cases} x(t) = e^{\frac{t}{2}}\left(A\sin 3t + B\cos 3t\right) \\ y(t) = e^{\frac{t}{2}}\left(C\sin 3t + D\cos 3t\right) \end{cases}$$

where A, B, C, and D are constants.

(i) Use the given form to find an expression for $x'(t)$.

(2 marks)

(ii) Use the initial conditions $x = 1$, $y = 0$ to find equations for A and B and hence find $x(t)$.

(3 marks)

(iii) Hence, or otherwise, find $y(t)$.

(3 marks)

(c) Calculate the height and width of the snail's shell after three periods of growth.

(4 marks)

Answer ***either*** *Question 15* ***or*** *Question 16.*

QUESTION 16 (15 marks)

The screenshot below is taken from an interactive table tennis game. The ball is following a curved path after being hit by the player in the foreground.

This photograph cannot be reproduced here for copyright reasons.

Source: *Wii Sports Resort*, Nintendo, 2009

The path of the ball can be modelled by placing axes on the screen and using a Bézier curve with the initial point $A(-1,\ -2)$, first control point $B(-1,\ 1)$, second control point $C(1,\ 4)$, and endpoint $D(2,\ 3)$. This is illustrated below.

This photograph cannot be reproduced here for copyright reasons.

Source: Adapted from *Wii Sports Resort*, Nintendo, 2009

The parametric equations for this curve are $\begin{cases} x(t) = -3t^3 + 6t^2 - 1 \\ y(t) = -4t^3 + 9t - 2 \end{cases}$ where $0 \le t \le 1$.

(a) (i) Given that t represents units of time, find the velocity vector $\boldsymbol{v}(t)$.

(2 marks)

(ii) Show that the maximum height of the ball's path occurs when $t = \frac{\sqrt{3}}{2}$ exactly.

(2 marks)

(iii) Find, correct to three significant figures, the coordinates of the point at which the height of the ball's path is a maximum.

(2 marks)

(b) (i) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t.

(1 mark)

(ii) Given that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{\frac{\mathrm{d}x}{\mathrm{d}t}}$, show that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{2\left(8t^2 - 9t + 6\right)}{3\left(3t^2 - 4t\right)^3}$.

(3 marks)

PLEASE TURN OVER

(c) (i) Table tennis players use spin to make the ball curve through the air. The curvature of a path is defined as

$$\kappa(t) = \frac{\left|\dfrac{d^2y}{dx^2}\right|}{\left(1+\left(\dfrac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}.$$

Find the curvature of the ball's path at maximum height.

(2 marks)

(ii) Find the point on the ball's path at which the curvature is a maximum.

(3 marks)

You may remove this page from the booklet by tearing along the perforations so that you can refer to it while you write your answers.

LIST OF MATHEMATICAL FORMULAE FOR USE IN STAGE 2 SPECIALIST MATHEMATICS

Circular Functions

$\sin^2 A + \cos^2 A = 1$

$\tan^2 A + 1 = \sec^2 A$

$1 + \cot^2 A = \operatorname{cosec}^2 A$

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$\sin 2A = 2 \sin A \cos A$

$\cos 2A = \cos^2 A - \sin^2 A$

$\quad = 2\cos^2 A - 1$

$\quad = 1 - 2\sin^2 A$

$\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$

$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$

$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$

$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$

$\sin A \pm \sin B = 2 \sin \frac{1}{2}(A \pm B) \cos \frac{1}{2}(A \mp B)$

$\cos A + \cos B = 2 \cos \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)$

$\cos A - \cos B = -2 \sin \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B)$

Matrices and Determinants

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\det A = |A| = ad - bc$ and

$A^{-1} = \dfrac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Derivatives

$f(x) = y$	$f'(x) = \dfrac{dy}{dx}$
x^n	nx^{n-1}
e^x	e^x
$\ln x = \log_e x$	$\dfrac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$

Properties of Derivatives

$\dfrac{d}{dx}\{f(x)\,g(x)\} = f'(x)g(x) + f(x)g'(x)$

$\dfrac{d}{dx}\left\{\dfrac{f(x)}{g(x)}\right\} = \dfrac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

$\dfrac{d}{dx} f(g(x)) = f'(g(x))\,g'(x)$

Quadratic Equations

If $ax^2 + bx + c = 0$ then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Distance from a Point to a Plane

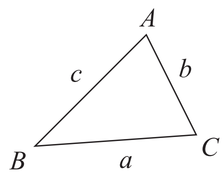
The distance from (x_1, y_1, z_1) to $Ax + By + Cz + D = 0$ is given by $\dfrac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$.

Mensuration

Area of sector $= \frac{1}{2} r^2 \theta$

Arc length $= r\theta$
(where θ is in radians)

In any triangle ABC:



Area of triangle $= \frac{1}{2} ab \sin C$

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

$a^2 = b^2 + c^2 - 2bc \cos A$