Government of South Australia

SACE Board of SA

External Examination 2009

2009 SPECIALIST MATHEMATICS

FOR OFFICE USE ONLY

SUPERVISOR CHECK

RE-MARKED

ATTACH SACE REGISTRATION NUMBER LABEL TO THIS BOX

Graphics calculator ☐

Brand ____________

Model ____________

Computer software ☐

Friday 13 November: 9 a.m.

Time: 3 hours

Pages: 43
Questions: 16

Examination material: one 43-page question booklet
one SACE registration number label

Approved dictionaries, notes, calculators, and computer software may be used.

Instructions to Students

1. You will have 10 minutes to read the paper. You must not write in your question booklet or use a calculator during this reading time but you may make notes on the scribbling paper provided.

2. This paper consists of three sections:

   Section A (Questions 1 to 10) 75 marks
   Answer ***all*** questions in Section A.

   Section B (Questions 11 to 14) 60 marks
   Answer ***all*** questions in Section B.

   Section C (Questions 15 and 16) 15 marks
   Answer ***one*** question from Section C.

3. Write your answers in the spaces provided in this question booklet. There is no need to fill all the space provided. You may write on pages 15, 32, and 42 if you need more space, making sure to label each answer clearly.

4. Appropriate steps of logic and correct answers are required for full marks.

5. Show all working in this booklet. (You are strongly advised ***not*** to use scribbling paper. Work that you consider incorrect should be crossed out with a single line.)

6. Use only black or blue pens for all work other than graphs and diagrams, for which you may use a sharp dark pencil.

7. State all answers correct to three significant figures, unless otherwise stated or as appropriate.

8. Diagrams, where given, are not necessarily drawn to scale.

9. The list of mathematical formulae is on page 43. You may remove the page from this booklet before the examination begins.

10. Complete the box on the top right-hand side of this page with information about the electronic technology you are using in this examination.

11. Attach your SACE registration number label to the box at the top of this page.

SECTION A (Questions 1 to 10)

(75 marks)

*Answer **all** questions in this section.*

QUESTION 1 (4 marks)

(a) Write $1+i\sqrt{3}$ exactly in $r\,\text{cis}\theta$ form, where $r>0$ and $-\pi<\theta\leq\pi$.

(2 marks)

(b) Hence evaluate $\left(1+i\sqrt{3}\right)^5$ in exact $a+ib$ form.

(2 marks)

QUESTION 2 (5 marks)

(a) Show that the line passing through the point $(1, -1, 2)$ and perpendicular to the plane $2x+3y-5z=8$ has the parametric equations

$$x = 1+2t$$
$$y = -1+3t$$
$$z = 2-5t$$

where t is any real number.

(2 marks)

(b) Find the point of intersection of the line and the plane in part (a).

(3 marks)

QUESTION 3 (8 marks)

Let $f(x) = \dfrac{4\sin x}{2+\cos x}$.

(a) Show that $f(x)$ is an odd function.

(2 marks)

(b) On the axes in Figure 1, sketch $y = f(x)$ for $-2\pi \le x \le 2\pi$.

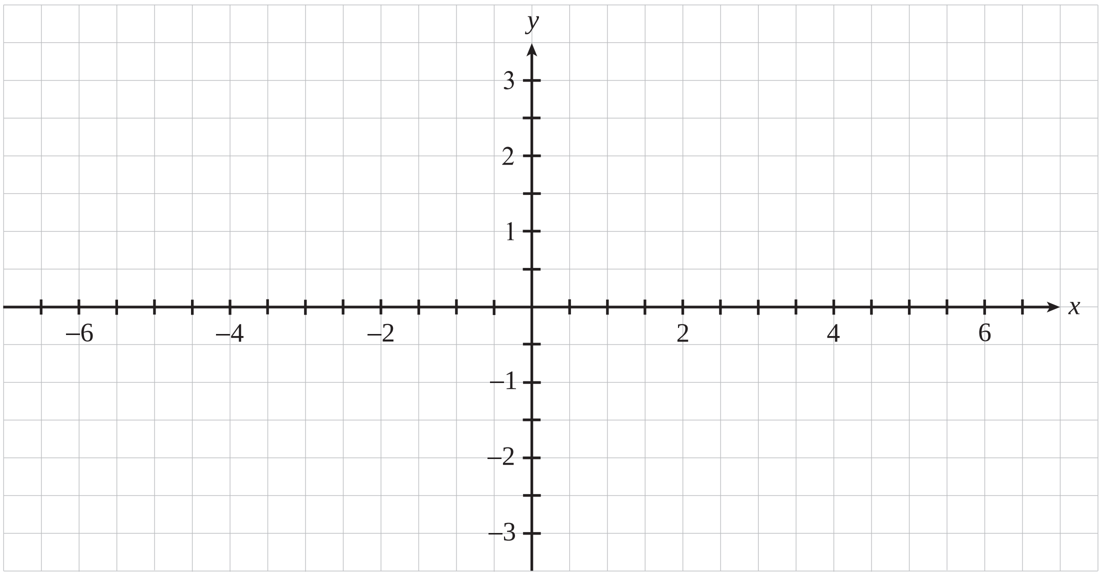


Figure 1 (3 marks)

(c) Find the area between the curve and the x-axis from $x = 0$ to $x = \dfrac{\pi}{4}$ in Figure 1.

(1 mark)

(d) For $-2\pi \leq k \leq 0$, give two exact values of k if $\int_{k}^{\frac{\pi}{4}} \frac{4\sin x}{2+\cos x}\,dx = 0$.

(2 marks)

PLEASE TURN OVER

QUESTION 4 (7 marks)

Figure 2 shows a circle with perpendicular chords AC and BD meeting at point P.
Line segment HP is perpendicular to BC, and $\angle BCA = \alpha$.
HP is extended to meet AD at point X.

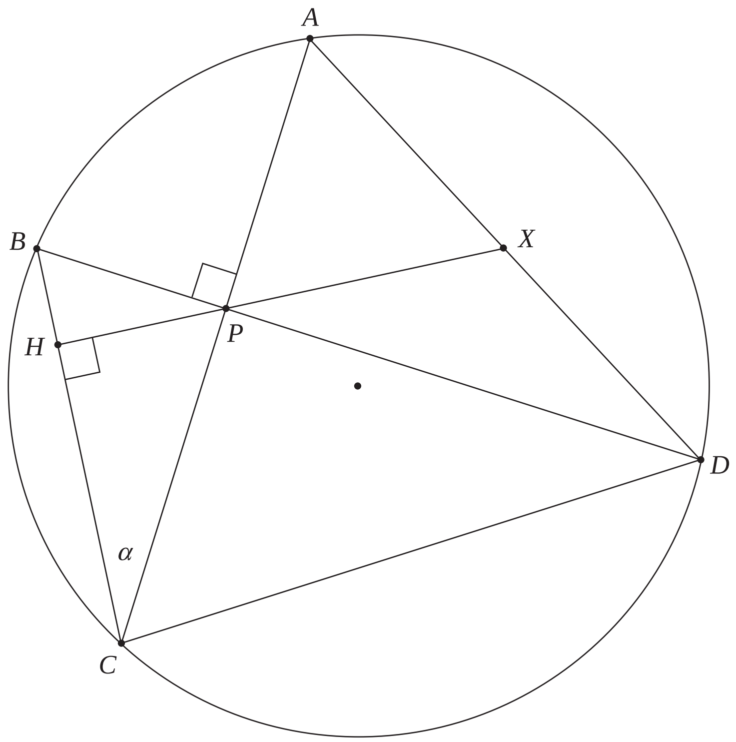


Figure 2

(a) Show that $\angle DPX = \angle BCA$.

(3 marks)

(b) Hence, or otherwise, show that triangle PAX is isosceles.

(2 marks)

(c) Hence show that X is the midpoint of AD.

(2 marks)

QUESTION 5 (6 marks)

Let $P(x) = x^4 + 6x^3 + 6x^2 - 22x - 36$.

(a) If $P(x)$ is divided by $x^2 - 4$, show that the quotient is $x^2 + 6x + 10$ and find the remainder.

(3 marks)

(b) Hence, or otherwise:

(i) find the remainder when $P(x)$ is divided by $(x - 2)$.

(2 marks)

(ii) show that $(x + 2)$ is a factor of $P(x)$.

(1 mark)

QUESTION 6 (7 marks)

Let $z = x + iy$ be a complex number such that $\dfrac{z-2i}{z-2}$ is purely imaginary.

(a) Show that $x^2 - 2x + y^2 - 2y = 0$.

(3 marks)

(b) Hence show that z lies on a circle with centre $1+i$ and radius $\sqrt{2}$.

(2 marks)

(c) Draw a diagram of the circle from part (b) and use it to find the largest value of $|z|$.

(2 marks)

QUESTION 7 (7 marks)

(a) Using the fact that $(r\,\text{cis}\theta)^{-1} = \dfrac{1}{r(\cos\theta + i\sin\theta)}$, prove that $(r\,\text{cis}\theta)^{-1} = r^{-1}\text{cis}(-\theta)$.

(2 marks)

(b) Using the fact that $(r\,\text{cis}\theta)^{-2} = (r\,\text{cis}\theta)^{-1}(r\,\text{cis}\theta)^{-1}$, prove that $(r\,\text{cis}\theta)^{-2} = r^{-2}\text{cis}(-2\theta)$.
(You may use $\text{cis}\alpha\,\text{cis}\beta = \text{cis}(\alpha + \beta)$.)

(2 marks)

(c) Use an inductive argument to show that De Moivre's theorem, $(r\,\text{cis}\,\theta)^n = r^n\text{cis}(n\theta)$, holds for all integers $n < 0$.

(3 marks)

PLEASE TURN OVER

QUESTION 8 (12 marks)

In Figure 3 line segment DB is a fixed diameter and point P is moving anticlockwise on a circle with centre O and a radius of r centimetres.

A tangent to the circle is drawn at P.

Point Q is the foot of the perpendicular from point D to the tangent.

Let $\angle BOP = \theta$, where $0 \le \theta \le \pi$.

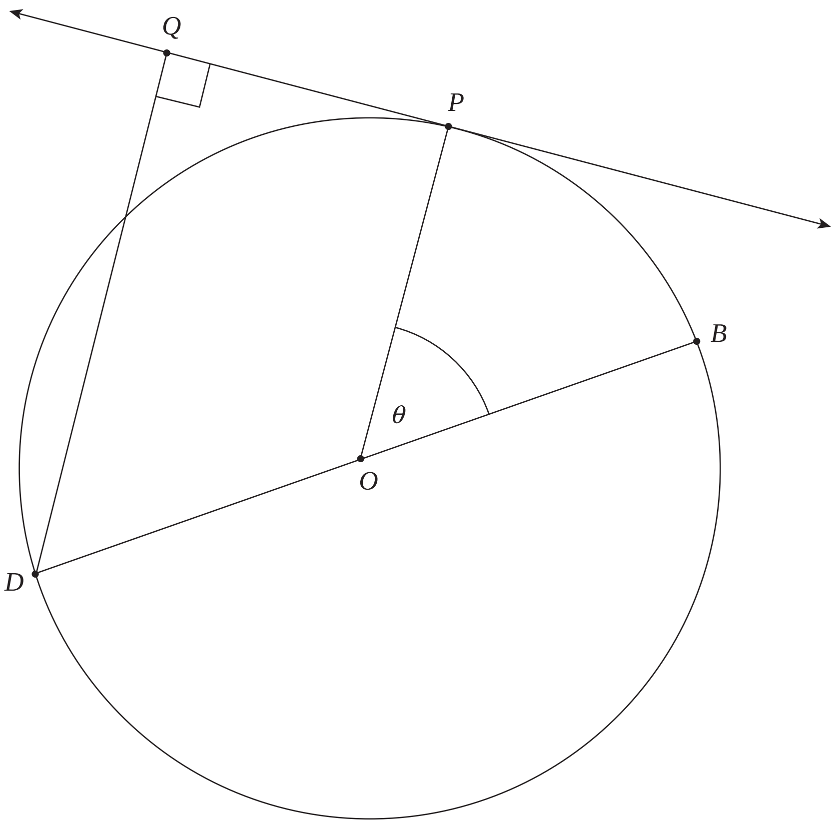


Figure 3

(a) Show that $DP^2 = 2r^2 + 2r^2 \cos\theta$.

(2 marks)

(b) (i) Show that DP bisects $\angle BDQ$.

(3 marks)

(ii) Hence show that $PQ = DP\sin\frac{\theta}{2}$ and $DQ = DP\cos\frac{\theta}{2}$.

(1 mark)

(iii) Hence, or otherwise, show that triangle DPQ has an area given by

$$A = \frac{1}{4}r^2\left(\sin 2\theta + 2\sin\theta\right).$$

(2 marks)

PLEASE TURN OVER

(c) As P moves anticlockwise around the circle, $\angle BOP = \theta$ increases at the rate of 5 radians per second.

Given that $r = 10$ centimetres, find the rate at which the area of triangle DPQ is changing at the instant when $\theta = \dfrac{\pi}{6}$.

(4 marks)

You may write on this page if you need more space to finish your answers. Make sure to label each answer carefully (e.g. 'Question 8(b)(ii) continued').

QUESTION 9 (10 marks)

Figure 4 shows the parallelogram *ABCD* where $\overrightarrow{AB} = \boldsymbol{a}$ and $\overrightarrow{BC} = \boldsymbol{b}$.

Point *X* divides *DB* internally in the ratio 2:1.

Point *M* is the midpoint of *AB*.

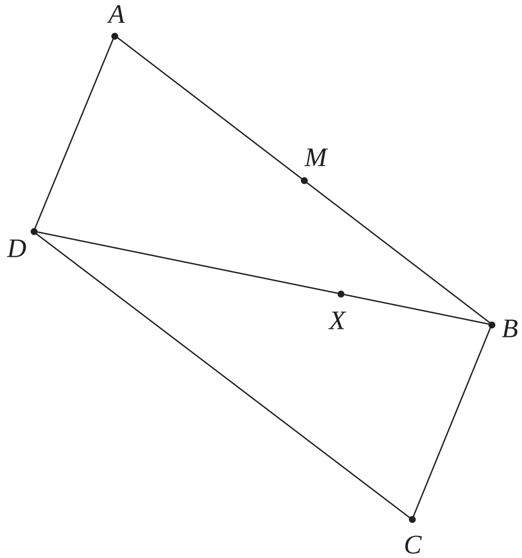


Figure 4

(a) Show that $\overrightarrow{DX} = \frac{2}{3}\boldsymbol{a} - \frac{2}{3}\boldsymbol{b}$.

(2 marks)

(b) Find $\overrightarrow{CX}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

(1 mark)

(c) Give a vector proof that points M, X, and C are collinear.

(3 marks)

(d) If $\boldsymbol{a} = [4,\ 2,\ -3]$ and $\boldsymbol{b} = [-1,\ 2,\ 2]$, find the area of triangle MXB.

(4 marks)

PLEASE TURN OVER

QUESTION 10 (9 marks)

Figure 5 shows a sequence of iterates in the interior of the Mandelbrot set.

The quadratic iteration is $z \to z^2 + c$, with $z_0 = 0$ and $c = \frac{-1}{5} + \frac{3}{5}i$.

This value of c is indicated by the cross in Figure 5. The other points show the sequence of iterates.

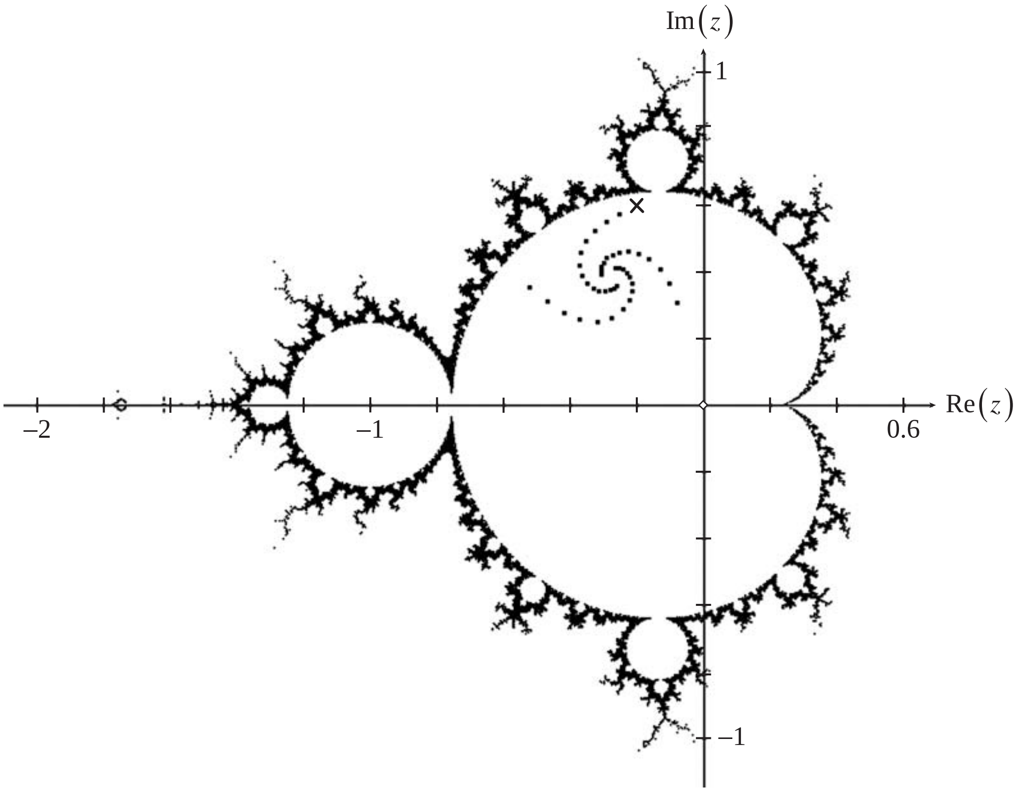


Figure 5

(a) Show that $|c| = \frac{\sqrt{10}}{5}$.

(1 mark)

(b) Given that $z_{n+1} = z_n^2 + c$, explain why $|z_{n+1}| \le |z_n|^2 + |c|$.

(2 marks)

(c) Given that $z_n = \frac{-6}{25} + \frac{2}{5}i$:

(i) calculate $|z_n|$, and without calculating $|z_{n+1}|$, use parts (a) and (b) to show that

$$|z_{n+1}| \le \frac{136 + 125\sqrt{10}}{625}.$$

(3 marks)

(ii) calculate z_{n+1} and hence verify the inequality found in part (c)(i).

(3 marks)

PLEASE TURN OVER

SECTION B (Questions 11 to 14)

(60 marks)

Answer ***all*** *questions in this section.*

QUESTION 11 (15 marks)

(a) Figure 6 shows point P on the circumference of a circle with diameter AB.

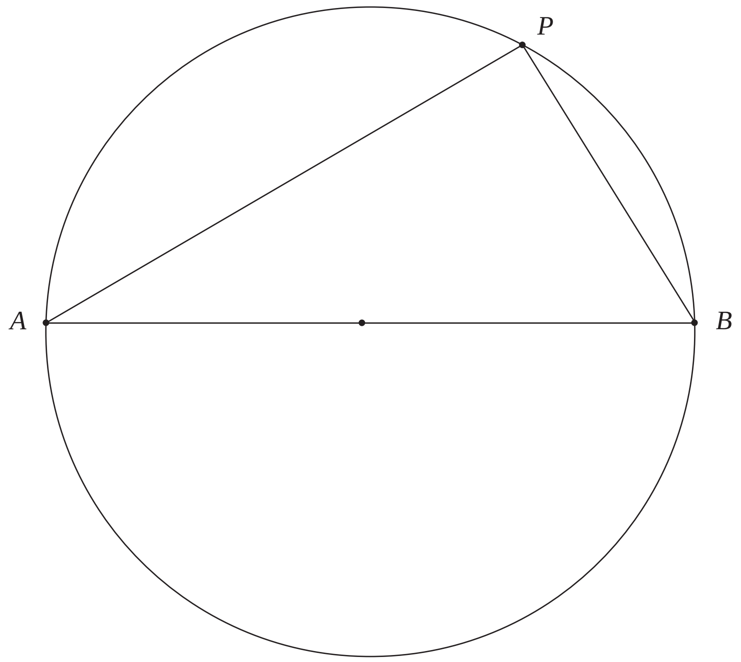


Figure 6

(i) Prove that $\overrightarrow{PA} \bullet \overrightarrow{PB} = 0$.

(1 mark)

(ii) If P is free to move on the circle between points A and B, explain why the maximum perpendicular distance of P from AB is half the length of the diameter.

(1 mark)

(b) If X is the point $(1+t,\ 2t,\ 3-t)$ where t is a parameter, A is the point $(0,\ -4,\ 2)$, and B is the point $(6,\ 8,\ -4)$:

(i) show that $\overrightarrow{XA} \bullet \overrightarrow{XB} = 6t^2 - 20t - 30$.

(2 marks)

(ii) find the coordinates of all points X, correct to three significant figures, such that $\angle AXB = 90°$.

(4 marks)

(c) (i) Show that AB is parallel to the line l_1 with parametric equations

$$x = 1+t,\ y = 2t,\ z = 3-t.$$

(2 marks)

(ii) Show that A, B, and l_1 are on the plane $3x - y + z = 6$.

(2 marks)

(iii) As shown in Figure 7, the line $l_2 : x = 8 + s,\ \ y = 4 + 2s,\ \ z = -14 - s$ is on the plane $3x - y + z = 6$.

Q is a point on l_2.

Is it possible for $\angle AQB$ to be a right angle? Explain.

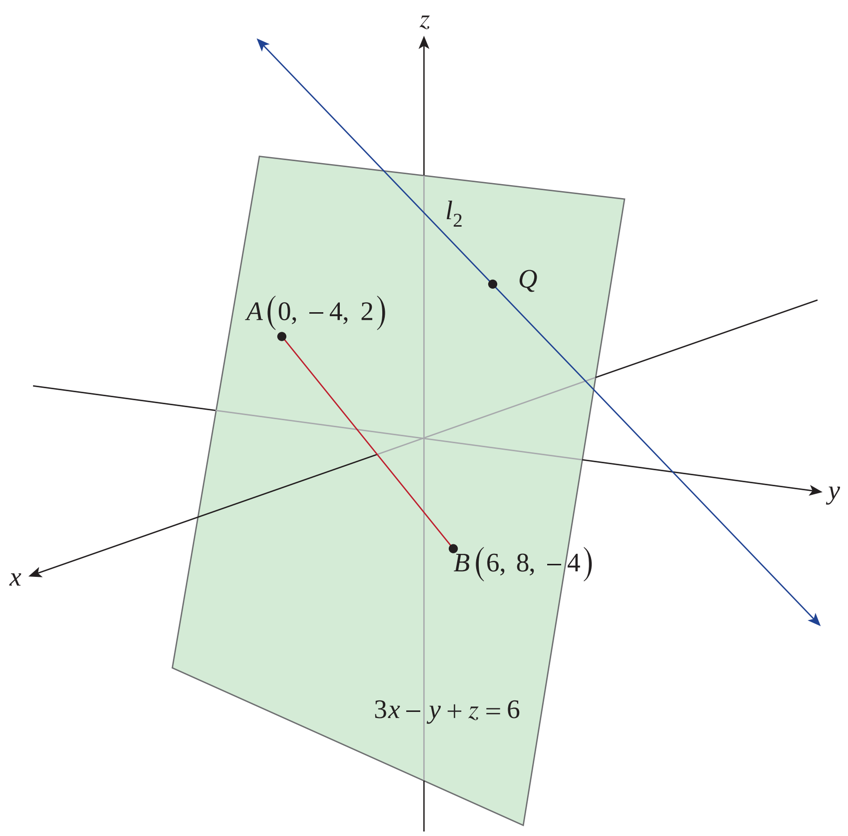


Figure 7

(3 marks)

QUESTION 12 (15 marks)

Let $y(t)$ be the weight in kilograms of a limb of an animal after t years.

The ratio $\dfrac{y'(t)}{y(t)}$ is called the relative growth rate of the limb.

The relative growth rate of the limb can be modelled by the differential equation

$$\frac{y'(t)}{y(t)} = \frac{k}{t}$$

where k, t, and y are all positive and k is a constant.

(a) The equations for Euler's method are

$$t_{n+1} = t_n + h$$
$$y_{n+1} = y_n + hy'(t_n).$$

(i) For the differential equation given above, show that $y_{n+1} = y_n\left(1 + \dfrac{hk}{t_n}\right)$.

(1 mark)

(ii) For the values $t_0 = 1$, $y_0 = 2$, and $h = 1$, use Euler's method to complete the table below in terms of k.

n	h	t_n	y_n	y_{n+1}
0		1	2	
1	1	2		
2	1	3		

(3 marks)

(b) By solving the differential equation $\frac{y'(t)}{y(t)} = \frac{k}{t}$, show that $y(t) = At^k$, where A is a constant.

(4 marks)

(c) If the weight of the limb is 2 kilograms after 1 year and 6 kilograms after 2 years, show that

$$y(t) = 2t^{\left(\frac{\ln 3}{\ln 2}\right)}.$$

(3 marks)

(d) Show that after 5 years the relative growth rate of the limb is given by

$$\frac{y'(5)}{y(5)} = \frac{\ln 3}{5\ln 2}.$$

(2 marks)

(e) After how many years is the relative growth rate of the limb less than 10%?

(2 marks)

PLEASE TURN OVER

QUESTION 13 (15 marks)

(a) Show that $(z-1)(z^4+z^3+z^2+z+1)=z^5-1$.

(1 mark)

(b) (i) Solve $z^5=1$, giving solutions in the form $r\operatorname{cis}\theta$.

(3 marks)

(ii) Draw the solutions for z on the Argand diagram in Figure 8.

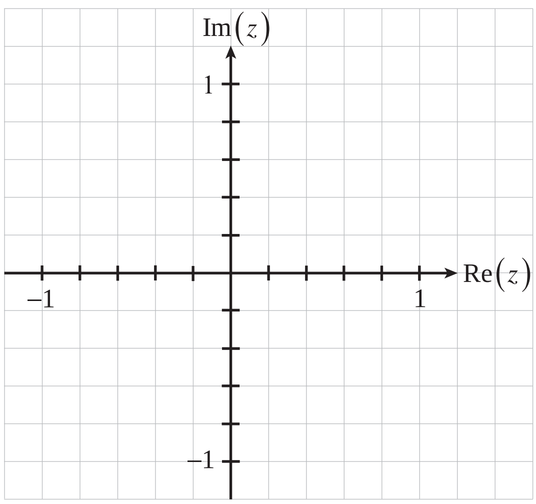


Figure 8

(2 marks)

(c) Let p and q be the roots of $z^5 - 1 = 0$ that have the two smallest positive arguments respectively.

(i) Show that $q = p^2$.

(1 mark)

(ii) Explain why the conjugate of p is p^{-1} and the conjugate of q is q^{-1}.

(2 marks)

(iii) Using conjugate pairs of roots and parts (c)(i) and (ii), show that

$$p^3\left(z^4 + z^3 + z^2 + z + 1\right) = \left(pz^2 - (q+1)z + p\right)\left(p^2z^2 - \left(q^2 + 1\right)z + p^2\right).$$

(4 marks)

PLEASE TURN OVER

(iv) To answer this part, use the following formula from part (c)(iii):

$$p^3\left(z^4+z^3+z^2+z+1\right)=\left(pz^2-(q+1)z+p\right)\left(p^2z^2-\left(q^2+1\right)z+p^2\right).$$

Let $z=-1$.

Show that $p+q+p^3+q^2=-1$.

(2 marks)

QUESTION 14 (15 marks)

Controlled by radio signals from Mission Command, an unmanned space probe is travelling in space in the region of a black hole. If the black hole is at the origin of the plane containing the space probe, Mission Command, and the black hole, the motion of the probe can be described by the differential system $\begin{cases} x' = 2x - 3y \\ y' = 3x - 4y \end{cases}$ where $(x(t),\ y(t))$ is the position of the probe at time t.

(a) Calculate the space probe's speed at the coordinates $(0,\ -6)$.

(2 marks)

(b) Figure 9 shows the slope field for the differential system given above.

On the figure, draw the solution curve that passes through $(0,\ -6)$.

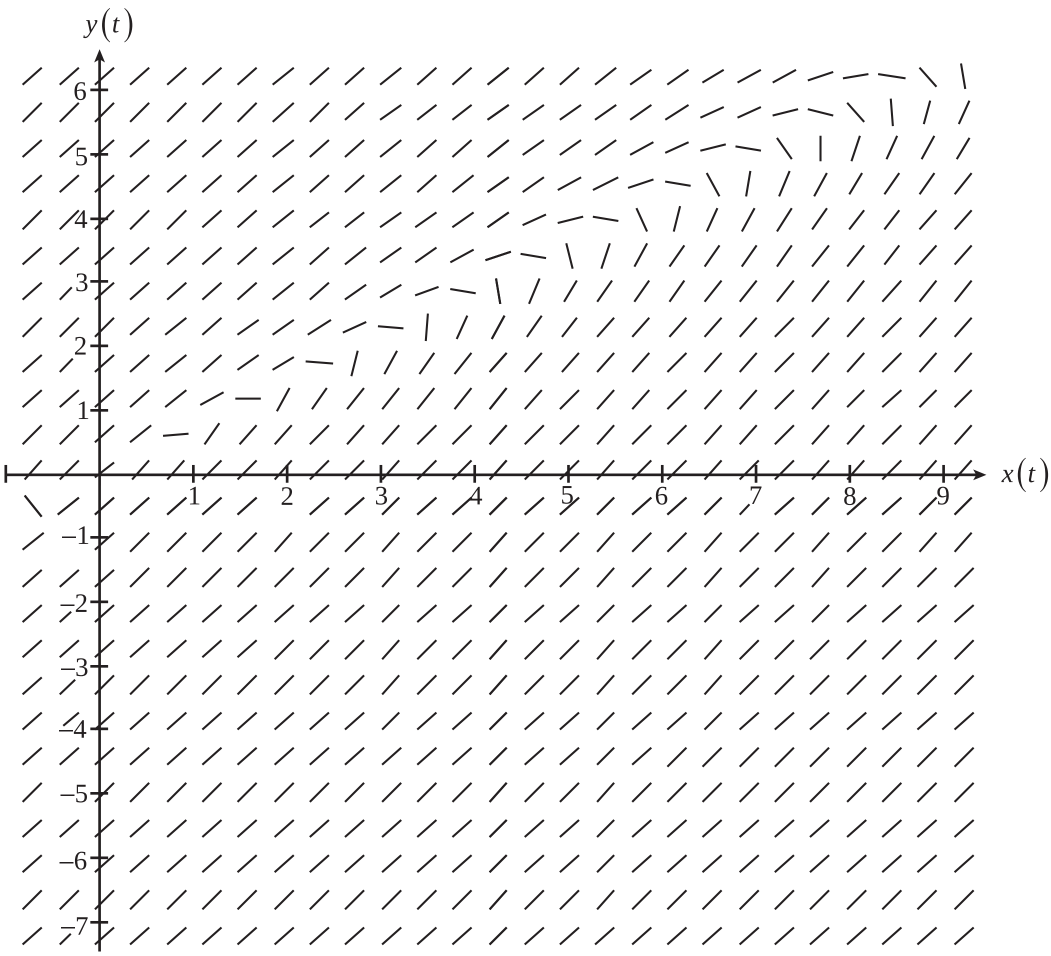


Figure 9

(3 marks)

PLEASE TURN OVER

The general solution for this differential system is of the form

$$\begin{cases} x(t) = (A + Bt)e^{-t} \\ y(t) = (C + Dt)e^{-t} \end{cases}$$

where A, B, C, and D are constants.

(c) Use this form to find $x'(t)$.

(1 mark)

(d) Hence use the initial conditions $x = 0,\ y = -6$ to find values for A and B.

(3 marks)

(e) Deduce the solution for $y(t)$.

(2 marks)

(f) Given that $\boldsymbol{P}(t)=[x(t),\ y(t)]$ and $\boldsymbol{v}(t)=[x',\ y']$ are the space probe's position and velocity vectors, discuss the probe's position and velocity as $t \to \infty$.

(4 marks)

You may write on this page if you need more space to finish your answers. Make sure to label each answer carefully (e.g. ‘Question 8(b)(ii) continued’).

SECTION C (Questions 15 and 16)

(15 marks)

*Answer **one** question from this section, **either** Question 15 **or** Question 16.*

*Answer **either** Question 15 **or** Question 16.*

QUESTION 15 (15 marks)

In a video soccer game animation, a player takes a free kick and sends the swerving, looping ball over and past the goalkeeper to hit the back of the net near the lower left-hand corner.

The path of the centre of the ball as it moves across the screen is shown in Figure 10. This path is modelled by the Bézier curve

$$x(t) = 20t^3 - 27t^2 - 3t + 5$$
$$y(t) = -12t^3 - 6t^2 + 27t - 6 \quad \text{where } 0 \le t \le 1.$$

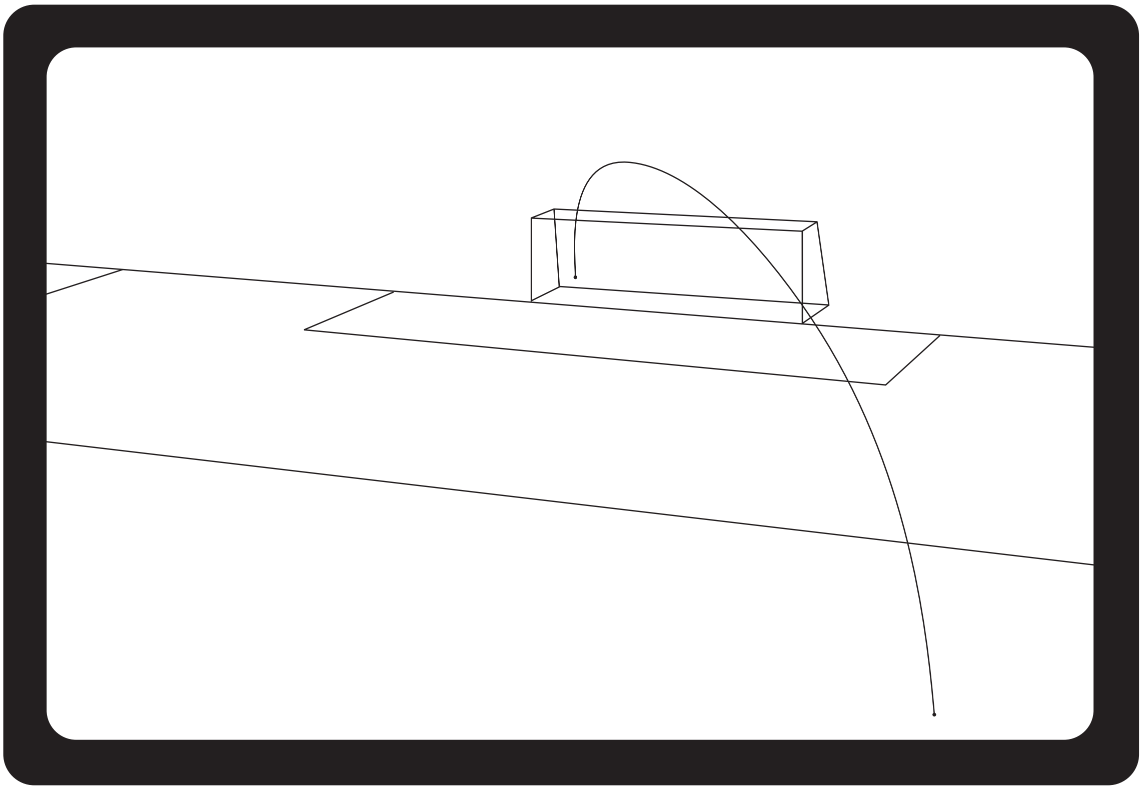


Figure 10

(a) Give the value of t and the coordinates of the centre of the ball (to three significant figures if appropriate) when:

(i) it is kicked.

(1 mark)

(ii) it is at the end of its path.

(1 mark)

(iii) it is at its highest point.

(3 marks)

(b) Let $t = 1$ represent 1 unit of time and $\boldsymbol{P}_t\left[x(t),\ y(t)\right]$ be the position vector of the ball's centre.

(i) Find the ball's velocity vector.

(2 marks)

(ii) Find the ball's maximum speed as it moves across the screen.

(3 marks)

(iii) If the screen distances are measured in centimetres and $t = 1$ corresponds to 0.46 seconds, give the ball's maximum speed in centimetres per second.

(1 mark)

(c) The length of a parametric curve $(x(t),\ y(t))$, where $a \le t \le b$, is calculated as the definite integral $\int_a^b \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$.

(i) Find the distance travelled by the ball as it moves across the screen.

(3 marks)

(ii) Hence find the average speed of the ball in centimetres per second as it moved across the screen.

(1 mark)

PLEASE TURN OVER

Answer ***either*** *Question 15* ***or*** *Question 16.*

QUESTION 16 (15 marks)

Information in nervous systems is transmitted between structures called neurons in the form of electrical pulses (see the illustration below). Each neuron receives electrical inputs from other neurons. When the inputs exceed a certain value, the neuron emits an electrical pulse and outputs it to other neurons.

The logistic function $y = \dfrac{1}{1+e^{-10x}}$ can be used to represent the electrical pulses through the process of iteration.

Source: www.lumosity.com/blog/your-nervous-system-at-work/

(a) On the axes in Figure 11, draw the graph of $y = \dfrac{1}{1+e^{-10x}}$.

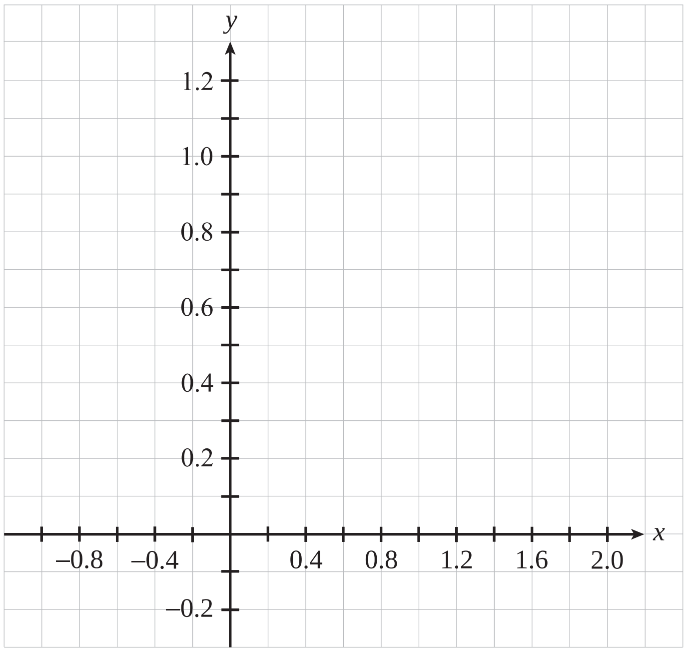


Figure 11 (3 marks)

A strong electrical pulse is said to occur when $y(x) = 1.00$ if rounded to two decimal places.
No electrical pulse is said to occur when $y(x) = 0.00$ if rounded to two decimal places.
Anything between these two values is called a weak electrical pulse.

Consider the iterative process $x_{t+1} = 0.2x_t - 3y(x_t) + 2,\ y_{t+1} = y(x_{t+1})$, where $x_0 = 0$.

$$\text{So, } x_1 = 0.2(0) - 3y(0) + 2$$
$$= 0 - 3\left(\frac{1}{2}\right) + 2$$
$$= \frac{1}{2}.$$
$$\text{Thus } y_1 = y\left(\frac{1}{2}\right)$$
$$= 0.9933...$$
$$= 0.99 \text{ (two decimal places).}$$

These values can be tabulated as follows:

t	x_t	$y_t = y(x_t)$	Electrical pulse
0	0	0.50*	weak
1	$\frac{1}{2}$	$0.9933... = 0.99*$	weak

* Two decimal places.

(b) (i) Carry out further iterations using $x_{t+1} = 0.2x_t - 3y(x_t) + 2,\ y_{t+1} = y(x_{t+1})$ to complete the table below.

t	x_t	$y_t = y(x_t)$	Electrical pulse
0	0	0.50*	weak
1	$\frac{1}{2}$	$0.9933... = 0.99*$	weak
2	$-0.8799...$		
3			
4			
5			
6			
7			

* Two decimal places.

(3 marks)

(ii) Describe the apparent behaviour of x_t (convergent, cyclic, or divergent).

(1 mark)

(iii) Describe the apparent behaviour of y_t (convergent, cyclic, or divergent).

(1 mark)

(c) A more general view can be taken by considering the logistic function

$$Y = \frac{1}{1+e^{-kx}}$$ where k is a positive constant.

(i) Complete the table below for $k = 1$.

t	x_t	$Y_t = Y(x_t)$	Electrical pulse
0	0	0.50	weak
1			
2			
3			
4			
5			
6			
7			

(2 marks)

(ii) Describe the apparent behaviour of Y_t for $k = 1$.

(1 mark)

(iii) Will Y_t ever produce a strong electrical pulse? Explain.

(2 marks)

(d) If $k = \ln\frac{49}{9}$, will the iterative function $Y = \frac{1}{1+e^{-kx}}$ ever produce a strong electrical pulse? Explain.

(2 marks)

You may write on this page if you need more space to finish your answers. Make sure to label each answer carefully (e.g. 'Question 8(b)(ii) continued').

You may remove this page from the booklet by tearing along the perforations so that you can refer to it while you write your answers.

LIST OF MATHEMATICAL FORMULAE FOR USE IN STAGE 2 SPECIALIST MATHEMATICS

Circular Functions

$\sin^2 A + \cos^2 A = 1$

$\tan^2 A + 1 = \sec^2 A$

$1 + \cot^2 A = \operatorname{cosec}^2 A$

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$\sin 2A = 2 \sin A \cos A$

$$\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A \\ &= 2\cos^2 A - 1 \\ &= 1 - 2\sin^2 A\end{aligned}$$

$\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$

$2 \sin A \cos B = \sin(A + B) + \sin(A - B)$

$2 \cos A \cos B = \cos(A + B) + \cos(A - B)$

$2 \sin A \sin B = \cos(A - B) - \cos(A + B)$

$\sin A \pm \sin B = 2 \sin \frac{1}{2}(A \pm B) \cos \frac{1}{2}(A \mp B)$

$\cos A + \cos B = 2 \cos \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)$

$\cos A - \cos B = -2 \sin \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B)$

Matrices and Determinants

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\det A = |A| = ad - bc$ and

$A^{-1} = \dfrac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Derivatives

$f(x) = y$	$f'(x) = \dfrac{dy}{dx}$
x^n	nx^{n-1}
e^x	e^x
$\ln x = \log_e x$	$\dfrac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$

Properties of Derivatives

$\dfrac{d}{dx}\{f(x)\,g(x)\} = f'(x)g(x) + f(x)g'(x)$

$\dfrac{d}{dx}\left\{\dfrac{f(x)}{g(x)}\right\} = \dfrac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

$\dfrac{d}{dx} f(g(x)) = f'(g(x))\,g'(x)$

Quadratic Equations

If $ax^2 + bx + c = 0$ then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Distance from a Point to a Plane

The distance from (x_1, y_1, z_1) to $Ax + By + Cz + D = 0$ is given by

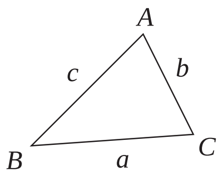
$\dfrac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$.

Mensuration

Area of sector $= \frac{1}{2} r^2 \theta$

Arc length $= r\theta$
(where θ is in radians)

In any triangle ABC:



Area of triangle $= \frac{1}{2} ab \sin C$

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

$a^2 = b^2 + c^2 - 2bc \cos A$