MA 1505 Mathematics I

Tutorial 9 Solutions

1. Use fundamental Theorem of line integral:

$$\int_C \nabla f \bullet d\mathbf{r} = f(\text{terminal point}) - f(\text{initial point}).$$

(a) C has parametric equation $\mathbf{r}(t) = (t^2+1)\mathbf{i} + (t^3+t)\mathbf{j}, \quad 0 \le t \le 1.$

So initial point is $\mathbf{r}(0) = (1,0)$ and terminal point is $\mathbf{r}(1) = (2,2)$.

So $\displaystyle\int_C \nabla f \bullet d\mathbf{r} = f(2,2) - f(1,0) = 9 - 3 \quad \text{(from table)} = 6.$

(b) The unit circle is a closed curve and ∇f is conservative. So $\displaystyle\oint_C \nabla f \bullet d\mathbf{r} = 0.$

2. Let C be the base of the fence which has vector equation

$$\mathbf{r}(t) = 10\cos t\mathbf{i} + 10\sin t\mathbf{j}, \quad 0 \le t \le 2\pi.$$

So $\|\mathbf{r}'(t)\| = 10$.

The area of the fence is then given by

$$\begin{aligned}\int_C h(x,y)ds &= \int_0^{2\pi} \left[4 + 0.01[(10\cos t)^2 - (10\sin t)^2]\right] \cdot 10\ dt \\ &= \int_0^{2\pi} 40 + 10\cos 2t\ dt \\ &= [40t + 5\sin 2t]_0^{2\pi} \\ &= 80\pi\end{aligned}$$

Both sides of the fence will give a total area of $160\pi = 503\ m^2$.

So the amount of paint used is about 5 litre.

3. The work done is given by the line integral $\int_C \mathbf{F} \bullet d\mathbf{r}$ where

$\mathbf{F}$ is the gravitational force contributed by the weight of the man + the pail of water;

C is the path traced out (which is a helix) by the man as he climbed up the staircase.

C has vector equation given by

$$\mathbf{r}(t) = 6\cos t\mathbf{i} + 6\sin(t)\mathbf{j} + \lambda t\mathbf{k}, \quad 0 \le t \le 6\pi$$

where λ is some constant.

As t increases from 0 to 6π (3 revolutions), the $\mathbf{k}$ component λt (representing the height) of C increases from 0 to 30.

i.e. $6\pi\lambda = 30$ or $\lambda = 5/\pi$. Hence

$$\mathbf{r}(t) = 6\cos t\mathbf{i} + 6\sin(t)\mathbf{j} + \frac{5}{\pi}t\mathbf{k} \quad 0 \le t \le 6\pi.$$

On the other hand, since $\mathbf{F}$ is given by gravitation, the vector field is of the form

$$\mathbf{F}(x,y,z) = 0\mathbf{i} + 0\mathbf{j} - (W_m + W_p)g\mathbf{k}$$

where W_m is the mass of the man and W_p is the mass of the pail of water.

Since the pail leaks 2kg of water throughout the ascent, W_p varies (linearly) according to z: As z increases from 0 to 30, W_p decreases from 10 to 8. i.e.

$$\frac{z-0}{W_p-10} = \frac{30-0}{8-10} \Longrightarrow z = -15(W_p - 10) \Longrightarrow W_p = 10 - \frac{z}{15}.$$

Hence

$$\mathbf{F}(x,y,z) = 0\mathbf{i} + 0\mathbf{j} - (90 - \frac{z}{15})g\mathbf{k}.$$

$$\begin{aligned}\int_C \mathbf{F} \bullet d\mathbf{r} &= \int_0^{6\pi} (0\mathbf{i} + 0\mathbf{j} - (90 - \frac{z}{15})g\mathbf{k}) \bullet (-6\sin t\mathbf{i} + 6\cos(t)\mathbf{j} + \frac{5}{\pi}\mathbf{k})\ dt \\ &= -\frac{5}{\pi}g\int_0^{6\pi} (90 - \frac{t}{3\pi})\ dt \\ &= -\frac{5}{\pi}g\left[90t - \frac{t^2}{6\pi}\right]_0^{6\pi} \\ &= -2670g\end{aligned}$$

So the work done is $2670g$ kg-m^{2}s^{-2} (against the gravity).

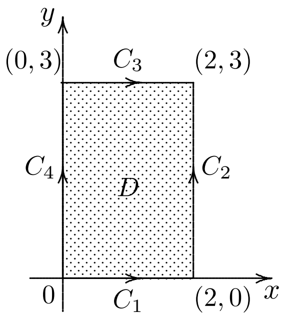
4. C is a piecewise smooth curves made up of 4 straight lines C_1, C_2, C_3, C_4.

Along C_1: $x = t$, $y = 0$
$\frac{dx}{dt} = 1$, $\frac{dy}{dt} = 0$, for $0 \le t \le 2$.

Along C_2: $x = 2$, $y = t$
$\frac{dx}{dt} = 0$, $\frac{dy}{dt} = 1$, for $0 \le t \le 3$.

Along C_3: $x = t$, $y = 3$
$\frac{dx}{dt} = 1$, $\frac{dy}{dt} = 0$, for $0 \le t \le 2$.

Along C_4: $x = 0$, $y = t$
$\frac{dx}{dt} = 0$, $\frac{dy}{dt} = 1$, for $0 \le t \le 3$.



If C is given positive orientation, then $C = C_1 + C_2 - C_3 - C_4$. Thus

$$\oint_C xy^2\,dx + x^3\,dy = \oint_{C_1+C_2-C_3-C_4} xy^2\,dx + x^3\,dy$$

$$= \int_0^2 0\,dt + \int_0^3 8\,dt - \int_0^2 9(t)\,dt - \int_0^3 0\,dt = 0+24-18-0 = 6.$$

We may also use Green's theorem to evaluate the line integral.

$$\oint_C xy^2\,dx + x^3\,dy = \iint_D \left[\frac{\partial}{\partial x}(x^3) - \frac{\partial}{\partial y}(xy^2)\right]\,dA$$

$$= \int_0^2 \int_0^3 (3x^2 - 2xy)\,dydx = \int_0^2 (9x^2 - 9x)\,dx = 6.$$

5. Let D be the ring region enclosed by two concentric circle of radius a and b respectively.

   In polar coordinates, $D: \quad a \le r \le b, \quad 0 \le \theta \le 2\pi$.

   By Green's Theorem

$$\oint_C (x^5 - y^5)\,dx + (x^5 + y^5)\,dy = \int\int_D \left[\frac{\partial}{\partial x}(x^5 + y^5) - \frac{\partial}{\partial y}(x^5 - y^5)\right]\,dA$$

$$= \int\int_D (5x^4 + 5y^4)\,dA = 5\int\int_D [(x^2+y^2)^2 - 2x^2y^2]\,dA$$

$$= 5\int_0^{2\pi}\int_a^b [(r^2)^2 - 2(r\cos\theta)^2(r\sin\theta)^2]r\,drd\theta = 5\int_0^{2\pi}\int_a^b r^5(1 - 2(\cos\theta)^2(\sin\theta)^2)\,drd\theta$$

$$= 5\left[\int_a^b r^5\,dr\right]\left[\int_0^{2\pi}(1 - 2(\cos\theta)^2(\sin\theta)^2)\,d\theta\right] = 5\left[\int_a^b r^5\,dr\right]\left[\int_0^{2\pi}(1 - \frac{1}{2}\sin^2 2\theta)\,d\theta\right]$$

$$= 5\left[\int_a^b r^5\,dr\right]\left[\int_0^{2\pi}\left(\frac{3}{4} + \frac{1}{4}\cos 4\theta\right)\,d\theta\right] = \left[\frac{5}{6}(b^6 - a^6)\right]\left[\frac{3}{2}\pi\right] = \frac{5}{4}\pi(b^6 - a^6).$$