MA 1505 Mathematics I

Tutorial 8 Solutions

1. Let $z = \sqrt{2^2 - x^2 - y^2}$. Then
   $z_x = -x(4 - x^2 - y^2)^{-1/2}$ and $z_y = -y(4 - x^2 - y^2)^{-1/2}$.
   Substitute $z = 1$ into $x^2 + y^2 + z^2 = 4$ gives

   $$x^2 + y^2 + 1 = 4 \quad \Rightarrow \quad x^2 + y^2 = 3$$

   which is the equation of a circle of radius $\sqrt{3}$.

   This means the plane $z = 1$ intersects the sphere at a circle of radius $\sqrt{3}$.

   Hence the projected region R of the part of the sphere is a disk of radius $\sqrt{3}$.

   In polar coordinates, this is given by

   $$0 \le r \le \sqrt{3}, \quad 0 \le \theta \le 2\pi.$$

   Thus,

   $$\begin{aligned} A(S) &= \iint_R \sqrt{\frac{x^2 + y^2}{4 - x^2 - y^2} + 1}\, dA = \int_0^{2\pi} \int_0^{\sqrt{3}} \left(\frac{r^2}{4 - r^2} + 1\right)^{\frac{1}{2}} r\, dr d\theta \\ &= \int_0^{2\pi} \int_0^{\sqrt{3}} 2r(4 - r^2)^{-\frac{1}{2}}\, dr d\theta = \int_0^{2\pi} d\theta \left[-2(4 - r^2)^{\frac{1}{2}}\right]_{r=0}^{r=\sqrt{3}} \\ &= (2\pi)[-2[(4 - 3)^{\frac{1}{2}} + 2[(4)^{\frac{1}{2}}] = 4\pi. \end{aligned}$$

2. We describe R as follows:

   $$R: \quad x \le y \le 2 - x^2, \quad -2 \le x \le 1.$$

   Then the mass of the lamina is

   $$\iint_R x^2\ dA = \int_{-2}^{1} \int_x^{2-x^2} x^2\ dy dx = \int_{-2}^{1} x^2(2 - x^2 - x)\ dx = \left[\frac{2}{3}x^3 - \frac{1}{5}x^5 - \frac{1}{4}x^4\right]_{-2}^{1} = \frac{63}{20}.$$

   Furthermore,

   $$\begin{aligned} \iint_R y(x^2)\ dA &= \int_{-2}^{1} \int_x^{2-x^2} yx^2\ dy dx = \int_{-2}^{1} \left[\frac{1}{2}x^2 y^2\right]_{y=x}^{y=2-x^2} dx \\ &= \frac{1}{2} \int_{-2}^{1} x^2(x^4 - 5x^2 + 4)\ dx = \frac{1}{2}\left[\frac{1}{7}x^7 - x^5 + \frac{4}{3}x^3\right]_{-2}^{1} = -\frac{9}{7}, \end{aligned}$$

   $$\begin{aligned} \iint_R x(x^2)\ dA &= \int_{-2}^{1} \int_x^{2-x^2} x^3\ dy dx = \int_{-2}^{1} x^3\left[(2 - x^2) - x\right]\ dx \\ &= \int_{-2}^{1} (2x^3 - x^5 - x^4)\ dx = \left[\frac{1}{2}x^4 - \frac{1}{6}x^6 - \frac{1}{5}x^5\right]_{-2}^{1} = -\frac{18}{5}, \end{aligned}$$

The required answer is

$$\bar{x} = \frac{-18/5}{63/20} = -\frac{8}{7}, \quad \bar{y} = \frac{-9/7}{63/20} = -\frac{20}{49}.$$

3. D is given by

$$-\frac{1}{2} \le x \le \frac{1}{2}, \quad -\frac{1}{2} \le y \le \frac{1}{2}, \quad -\frac{1}{2} \le z \le \frac{1}{2}$$

So

$$\begin{aligned}
\iiint_D (x^2 + 2z)dV &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} (x^2 + 2z)dx\ dz\ dy \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[\frac{1}{3}x^3 + 2xz\right]_{-\frac{1}{2}}^{\frac{1}{2}} dz\ dy \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} (\frac{1}{12} + 2z)dz\ dy \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[\frac{z}{12} + z^2\right]_{-\frac{1}{2}}^{\frac{1}{2}} dy \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{12}\ dy = \frac{1}{12}
\end{aligned}$$

4. We use the criteria of conservative field:

$$\begin{aligned}
\frac{\partial}{\partial y}(2xy) = 2x &= \frac{\partial}{\partial x}(x^2 + 2yz), \\
\frac{\partial}{\partial z}(2xy) = 0 &= \frac{\partial}{\partial x}(y^2), \\
\frac{\partial}{\partial z}(x^2 + 2yz) = 2y &= \frac{\partial}{\partial y}(y^2).
\end{aligned}$$

So $\mathbf{F}$ is conservative, and hence there exists a function f such that $\nabla f = \mathbf{F}$.

To find f, first we know that, by $\mathbf{i}$-component of $\mathbf{F}$, we have $f_x(x, y, z) = 2xy$ so that

$$f(x, y, z) = x^2 y + g(y, z) \quad (*).$$

Differentiate $(*)$ w.r.t. y, we have

$$f_y(x, y, z) = x^2 + g_y(y, z).$$

Now by $\mathbf{j}$-component of $\mathbf{F}$, we have $f_y(x, y, z) = x^2 + 2yz$, so

$$g_y(y, z) = 2yz \Rightarrow g(y, z) = y^2 z + h(z).$$

Hence $(*)$ becomes

$$f(x,y,z) = x^2y + y^2z + h(z) - -(**)$$

Differentiate $(**)$ w.r.t. z, we have $f_z(x,y,z) = y^2 + h'(z)$.
Now by **k**-component of **F**, $f_z(x,y,z) = y^2$ so

$$h'(z) = 0 \Rightarrow h(z) = K.$$

So $f(x,y,z) = x^2y + y^2z + K$, where K is a constant.

5. $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} + 3t\mathbf{k}$, where $0 \le t \le 1$. Then

$$\mathbf{r}'(t) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}, \quad ||\mathbf{r}'(t)|| = \sqrt{14}, \quad g(x(t), y(t), z(t)) = t^2 - (2t)(3t) + (3t)^2 = 4t^2.$$

Therefore

$$\int_C g(x,y,z)\ ds = \int_0^1 (4t^2)(\sqrt{14})\ dt = \frac{4}{3}\sqrt{14}.$$

6. $\mathbf{F}(\mathbf{r}(t)) = t^5\mathbf{i} + 2t^2\mathbf{j} - t^2\mathbf{k}$. $\mathbf{r}'(t) = 1\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}$.

$$\begin{aligned}\int_C \mathbf{F} \bullet d\mathbf{r} &= \int_0^1 (t^5\mathbf{i} + 2t^2\mathbf{j} - t^2\mathbf{k}) \bullet (1\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k})\, dt \\ &= \int_0^1 (t^5 - 3t^4 + 4t^3)\, dt = \frac{17}{30}.\end{aligned}$$

7. To parametrize line segment from (a,b,c) to (d,e,f), we can use

$$\mathbf{r}(t) = [a\mathbf{i} + b\mathbf{j} + c\mathbf{k}] + t[(d\mathbf{i} + e\mathbf{j} + f\mathbf{k}) - (a\mathbf{i} + b\mathbf{j} + c\mathbf{k})], \quad 0 \le t \le 1.$$

So we have
the line segment C_1 joining $(0,0,0)$ to $(1,0,2)$

$$\mathbf{r}(t) = t\mathbf{i} + 0\mathbf{j} + 2t\mathbf{k}, \quad 0 \le t \le 1$$

the line segment C_2 joining $(1,0,2)$ to $(3,4,1)$

$$\mathbf{r}(t) = (2t+1)\mathbf{i} + 4t\mathbf{j} + (-t+2)\mathbf{k}, \quad 0 \le t \le 1$$

Then

$$\int_{C_1} 2xy\ dx + (x^2 + z)\ dy + y\ dz = 0,$$

$$\int_{C_2} 2xy\ dx + (x^2 + z)\ dy + y\ dz$$

$$= \int_0^1 2(2t+1)(4t)(2dt) + ((2t+1)^2 + (-t+2))(4dt) + (4t)(-dt)$$

$$= \int_0^1 (48t^2 + 24t + 12)dt = 40.$$

So $\displaystyle\int_C 2xy\ dx + (x^2 + z)\ dy + y\ dz = 0 + 40 = 40.$