MA 1505 Mathematics I

Tutorial 7 Solutions

1. (a) $$\int_0^b \int_0^a (x^2+y^2)\,dxdy = \int_0^b \left[\frac{1}{3}x^3 + xy^2\right]_{x=0}^{x=a} dy = \int_0^b \left(\frac{1}{3}a^3 + ay^2\right) dy$$
$$= \left[\frac{1}{3}a^3y + \frac{1}{3}ay^3\right]_0^b = \frac{1}{3}a^3b + \frac{1}{3}ab^3.$$

(b) $$\int_1^2 \int_0^1 \frac{xy}{\sqrt{4-x^2}}\,dxdy = \int_1^2 \left[-\frac{1}{2}y\left(2(4-x^2)^{1/2}\right)\right]_{x=0}^{x=1} dy$$
$$= \int_1^2 -y(3^{1/2}-4^{1/2})\,dy$$
$$= (2-\sqrt{3})\left[\frac{1}{2}y^2\right]_{y=1}^{y=2} = 3 - \frac{3}{2}\sqrt{3}.$$

2. (a) The region can be regarded as a Type A region

$$D: \quad 0 \le y \le x, \quad 0 \le x \le 1.$$

$$\int_0^1 \int_0^x e^{x^2}\,dydx = \int_0^1 \left[ye^{x^2}\right]_{y=0}^{y=x} dx = \int_0^1 xe^{x^2}\,dx$$
$$= \frac{1}{2}\left[e^{x^2}\right]_0^1 = \frac{1}{2}(e-1).$$

(b) The region can be regarded as a type A region with bottom boundary $y = x^2$ and top boundary $y = \sqrt{x}$.
Since the two curves intersect at $x = 0$ and $x = 1$, the left and right are bounded by $x = 0$ and $x = 1$ respectively. So

$$D: \quad x^2 \le y \le \sqrt{x}, \quad 0 \le x \le 1.$$

$$\int_0^1 \int_{x^2}^{\sqrt{x}} (x+y)\,dydx = \int_0^1 \left[xy + \frac{1}{2}y^2\right]_{y=x^2}^{y=\sqrt{x}} dx = \int_0^1 (x^{3/2} + \frac{1}{2}x - x^3 - \frac{1}{2}x^4)\,dx$$
$$= \left[\frac{1}{5}2x^{5/2} + \frac{1}{4}x^2 - \frac{1}{4}x^4 - \frac{1}{10}x^5\right]_0^1 = \frac{3}{10}.$$

3. The line joining $(1,0)$ and $(4,2)$ has equation

$$\frac{y-0}{x-1} = \frac{2-0}{4-1} = \frac{2}{3} \iff y = \frac{2}{3}x - \frac{2}{3} \iff x = \frac{3}{2}y + 1.$$

The line joining $(1,0)$ and $(9,-3)$ has equation

$$\frac{y-0}{x-1} = \frac{(-3)-0}{9-1} = -\frac{3}{8} \iff y = -\frac{3}{8}x + \frac{3}{8} \iff x = -\frac{8}{3}y + 1.$$

The region $D$ is is the union of $D_1$ and $D_2$, where

$$D_1: \quad y^2 \le x \le \frac{3}{2}y + 1, \quad 0 \le y \le 2,$$
$$D_2: \quad y^2 \le x \le -\frac{8}{3}y + 1, \quad -3 \le y \le 0.$$

Hence the required answer is

$$\begin{aligned}\iint_D x\,dA &= \iint_{D_1} x\,dA + \iint_{D_2} x\,dA \\ &= \int_0^2 \int_{y^2}^{(3y/2)+1} x\,dxdy + \int_{-3}^0 \int_{y^2}^{-(8y/3)+1} x\,dxdy \\ &= \frac{19}{5} + \frac{106}{5} = 25,\end{aligned}$$

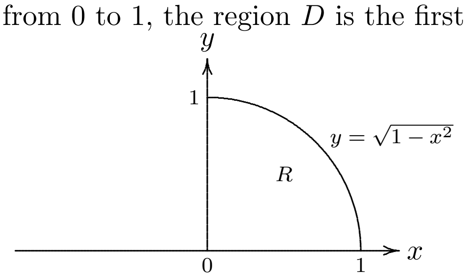
since

$$\int_0^2 \int_{y^2}^{(3y/2)+1} x\,dxdy = \int_0^2 \frac{1}{8}(9y^2 + 12y + 4 - 4y^4)\,dy = \frac{19}{5},$$
$$\int_{-3}^0 \int_{y^2}^{-(8y/3)+1} x\,dxdy = \int_{-3}^0 \frac{1}{18}(64y^2 - 48y + 9 - 9y^4)\,dy = \frac{106}{5}.$$

4. The region in Cartesian coordinates is given by

$$D: \quad 0 \le y \le \sqrt{1-x^2}, \quad 0 \le x \le 1$$

This is a type A region with $x$-axis as the bottom boundary and upper half of the unit circle as the upper boundary.
Since the range of $x$ is from 0 to 1, the region $D$ is the first quadrant of the unit disk.




In polar coordinates, this is given by

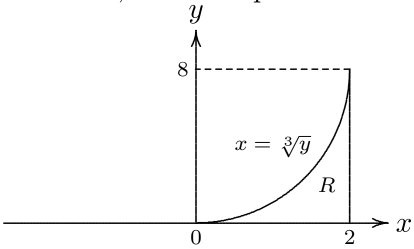
$$D: \quad 0 \le r \le 1, \quad 0 \le \theta \le \pi/2.$$

$$\begin{aligned}\int_0^1 \int_0^{\sqrt{1-x^2}} e^{x^2+y^2}\,dydx &= \int_0^{\pi/2} \int_0^1 e^{r^2} r\,drd\theta = \int_0^{\pi/2} d\theta \int_0^1 e^{r^2} r\,dr \\ &= \frac{\pi}{2}\left[\frac{1}{2}e^{r^2}\right]_0^1 = \tfrac{1}{4}\pi(e-1).\end{aligned}$$

5. (a) The type B region $R$ is given by

$$\sqrt[3]{y} \le x \le 2, \quad 0 \le y \le 8.$$

It is bounded on the left by the cubic curve $\sqrt[3]{y} = x$ and on the right by the vertical line $x = 2$.
Below it is bounded by the $x$-axis, and on top the left and right boundaries intersect at $y = 8$.




Converting to type A region, the lower boundary is $y = 0$, the top boundary is the cubic curve $y = x^3$.
On the left, these two boundaries intersect at $x = 0$ and on the right, it is bounded by $x = 2$.
So the region is given by

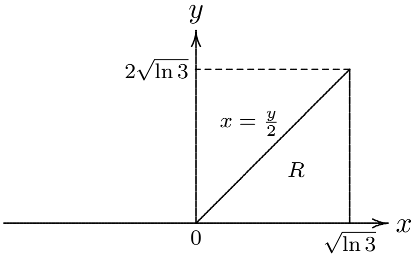
$$0 \le y \le x^3, \quad 0 \le x \le 2.$$

$$\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\, dxdy = \int_0^2 \int_0^{x^3} e^{x^4}\, dydx = \int_0^2 e^{x^4} [y]_{y=0}^{y=x^3}\, dx = \int_0^2 x^3 e^{x^4}\, dx$$
$$= \left[\frac{1}{4}e^{x^4}\right]_0^2 = \frac{1}{4}(e^{16} - 1).$$

(b) The type B region $R$ is given by

$$y/2 \le x \le \sqrt{\ln 3}, \quad 0 \le y \le 2\sqrt{\ln 3}.$$

It is bounded on the left by the straight line $x = y/2$ and on the right by the vertical line $x = \sqrt{\ln 3}$.
Below it is bounded by the $x$-axis, and on top the left and right boundaries intersect at $y = 2\sqrt{\ln 3}$.




Converting to type A region, the lower boundary is $y = 0$, the top boundary is the line $y = 2x$.
On the left, these two boundaries intersect at $x = 0$ and on the right, it is bounded by $x = \sqrt{\ln 3}$.
So the region is given by

$$0 \le y \le 2x, \quad 0 \le x \le \sqrt{\ln 3}.$$

$$\int_0^{2\sqrt{\ln 3}} \int_{y/2}^{\sqrt{\ln 3}} e^{x^2}\, dxdy = \int_0^{\sqrt{\ln 3}} \int_0^{2x} e^{x^2}\, dydx = \int_0^{\sqrt{\ln 3}} e^{x^2} [y]_{y=0}^{y=2x}\, dx = \int_0^{\sqrt{\ln 3}} 2xe^{x^2}\, dx$$
$$= \left[e^{x^2}\right]_0^{\sqrt{\ln 3}} = e^{\ln 3} - 1 = 2.$$

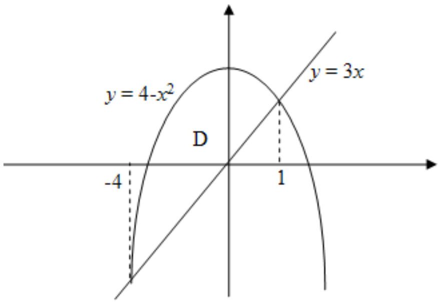
6. The volume is given by the double integral

$$V = \iint_D f(x,y)dA$$

where $D$ is the region bounded by the parabola $y = 4 - x^2$ and straight line $y = 3x$ and $f(x,y)$ is the function whose graph is the plane $x - z + 4 = 0$.

Writing the equation of the plane as $z = x + 4$, we get the function $f(x,y) = x + 4$.

A rough sketch of the region $D$ is shown below:




$D$ can be regarded as type A region

$$D: \quad 3x \le y \le 4 - x^2, \quad -4 \le x \le 1.$$

(The two limits $-4$ and $1$ of $x$ are obtained by solving the two equation $y = 3x$ and $y = 4 - x^2$.)

Hence

$$V = \int_{-4}^{1} \int_{3x}^{4-x^2} (x+4)dydx = \int_{-4}^{1} (x+4)(4-x^2-3x)dx = \left[16x - 4x^2 - \frac{7}{3}x^3 - \frac{1}{4}x^4\right]_{-4}^{1} = \frac{625}{12}$$

7. By symmetry, the desired volume $V$ is 8 times the volume $V_1$ in the first octant. Now,

$$V_1 = \int_0^r \int_0^{\sqrt{r^2-y^2}} \sqrt{r^2-y^2}\, dxdy = \int_0^r \left[x\sqrt{r^2-y^2}\right]_{x=0}^{x=\sqrt{r^2-y^2}} dy$$
$$= \int_0^r (r^2 - y^2)\, dy = \left[r^2 y - \frac{1}{3}y^3\right]_0^r = \frac{2}{3}r^3.$$

Therefore, $V = \frac{16}{3}r^3$.