MA1506 Tutorial 1 Solutions

(1a)
$$y' = \frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1} \rightarrow y = \ln\left|\frac{x}{x+1}\right| + c$$

(1b)
$$y' = \cos x \cos 5x = \frac{1}{2}[\cos 6x + \cos 4x] \rightarrow y = \frac{1}{2}\left[\frac{1}{6}\sin 6x + \frac{1}{4}\sin 4x\right] + c$$

(1c)
$$\frac{dy}{dx} = e^x e^{-3y} \Rightarrow e^{3y}dy = e^x dx \Rightarrow \frac{1}{3}e^{3y} = e^x + c$$

(1d)
$$\frac{1+y}{y^2}dy = (2x-1)dx \rightarrow \ln|y| - \frac{1}{y} = x^2 - x + c$$

(2)
$\frac{dT}{dt} = -k(T - Tenv)$ where k is a positive constant. (If k were negative, then hot objects would get hotter when left to "cool". That doesn't happen – if it did, we would not be here to discuss it.) $T = Tenv$ is obviously a solution since $\frac{dTenv}{dt} = 0$ and this does make sense because objects do not spontaneously become hotter or colder. Having settled this case, we can assume $T \neq Tenv$ and so we can write
$\frac{dT}{T - Tenv} = -kdt$ so $\ln|T - Tenv| = -kt + c$.

In the case at hand, $T > Tenv$ so $|T - Tenv| = T - Tenv$ and so $T = Tenv + \alpha e^{-kt}$.

At $t = 0$, $T = 300$ so $300 = 75 + \alpha \Rightarrow \alpha = 225$. At $t = \frac{1}{2}$, $T = 200$, so

$$200 = 75 + 225e^{-k/2} \Rightarrow k = -2\ln\frac{125}{225} = 1.1756$$

Thus $T(3) = 75 + 225e^{-3k} \approx 81.6$.

(3)
The volume V is related to the area A by V = a $A^{(3/2)}$ where a is a positive constant with no units; this is reasonable because volume has units of cubic metres and area has units of square metres. [Of course, ``reasonable" doesn't mean that it's always exactly true.] Then dV/dt = (3a/2)(dA/dt) $A^{(1/2)}$

The question tells us that dV/dt = - bA, where b is a positive constant with units of metres/sec. This is reasonable because evaporation takes place at the surface of the drop and so its rate can be expected to depend on the area. So we have $dA/ A^{(1/2)}$ =

- 2bdt/3a. Integrating from A_0, the initial area, up to zero, we find that the time taken for complete evaporation is $3aA_0^{(1/2)}/b$ which does indeed have units of time since the numerator has units of metres while the denominator has units of metres/sec.

If, instead of dV/dt = - bA, we propose that $dV/dt = - bA^2$, then we would have obtained $dA/ A^{(3/2)}$ = - 2bdt/3a . When you try to integrate this from A_0 to zero, you will get a divergent integral, meaning that the evaporation would take infinite time and the rain would always reach the ground, contrary to the definition of Virga.

(4)

The moth flies in such a way that the angle ψ remains constant at all times, so we have a differential equation tan (ψ) = constant = rdθ/dr, hence dr/r = dθ / tan (ψ) , thus r = R exp(θ / tan (ψ)), where we take it that θ = 0 when the moth first sees the candle, and that her distance from the candle is R at that time. Remember that ψ is the angle between the radius vector of the moth [pointing outwards] and her velocity. From the point of view of the moth looking towards a candle in front of her, ψ would be an angle greater than 90 degrees. Draw a diagram if this is not obvious! Thus tan (ψ) will be negative and r will get steadily smaller as θ increases. Such a curve is called a spiral. So the unfortunate moth will spiral into the candle with tragic consequences. Of course if her first view of the candle is over her ``shoulder" then tan (ψ) will be positive and she will spiral outwards, something that would be a lot less noticeable. Finally if ψ is 90 degrees exactly, the moth will fly along a circle until it drops dead from exhaustion or starvation, whichever comes first.

(6)

These are examples of ODEs where a change of variable is needed.

(6a)

Let $v = 2x + y$ so $v' = 2 + y'$

$$\Rightarrow v' - 2 = \frac{1-2v}{1+v} \Rightarrow v' = \frac{3}{1+v}$$

$$\Rightarrow v + \frac{1}{2}v^2 = 3x + c$$

$$\Rightarrow (2x+y) + \frac{1}{2}(2x+y)^2 = 3x + c$$

(6b)

$v = x + y$ $\qquad$ $y' = v' - 1 = \left(\frac{v+1}{v+3}\right)^2$

$$\Rightarrow v' = 1 + \frac{v^2 + 2v + 1}{(v+3)^2} = \frac{2v^2 + 8v + 10}{(v+3)^2}$$

$$\Rightarrow \frac{(v+3)^2}{v^2 + 4v + 5} dv = 2dx$$

$$\Rightarrow \frac{v^2 + 4v + 5 + 2v + 4}{v^2 + 4v + 5} dv = 2dx$$

$$\Rightarrow \left(1 + \frac{2v + 4}{v^2 + 4v + 5}\right) dv = 2dx$$

$$\Rightarrow v + \ln\left|\left(v^2 + 4v + 5\right)\right| = 2x + c$$

$$\Rightarrow x + y + \ln\left|(x+y)^2 + 4x + 4y + 5\right| = 2x + c$$