Government of South Australia

SACE Board of SA

External Examination 2009

2009 MATHEMATICAL STUDIES

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ATTACH SACE REGISTRATION NUMBER LABEL TO THIS BOX

Graphics calculator ☐

Brand ______________

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Thursday 5 November: 9 a.m.

Time: 3 hours

Pages: 37
Questions: 16

Examination material: one 37-page question booklet
one SACE registration number label

Approved dictionaries, notes, calculators, and computer software may be used.

Instructions to Students

1. You will have 10 minutes to read the paper. You must not write in your question booklet or use a calculator during this reading time but you may make notes on the scribbling paper provided.

2. Answer ***all*** parts of Questions 1 to 16 in the spaces provided in this question booklet. There is no need to fill all the space provided. You may write on pages 11, 23, 29, and 35 if you need more space, making sure to label each answer clearly.

3. The total mark is approximately 143. The allocation of marks is shown below:

Question	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Marks	8	6	7	4	8	8	4	11	9	9	9	10	12	11	11	16

4. Appropriate steps of logic and correct answers are required for full marks.

5. Show all working in this booklet. (You are strongly advised ***not*** to use scribbling paper. Work that you consider incorrect should be crossed out with a single line.)

6. Use only black or blue pens for all work other than graphs and diagrams, for which you may use a sharp dark pencil.

7. State all answers correct to three significant figures, unless otherwise stated or as appropriate.

8. Diagrams, where given, are not necessarily drawn to scale.

9. The list of mathematical formulae is on page 37. You may remove the page from this booklet before the examination begins.

10. Complete the box on the top right-hand side of this page with information about the electronic technology you are using in this examination.

11. Attach your SACE registration number label to the box at the top of this page.

QUESTION 1

(a) Find $\frac{dy}{dx}$ for each of the following functions. There is no need to simplify your answers.

(i) $y = \frac{3}{x^4} - 10\ln x$.

(2 marks)

(ii) $y = \frac{8x - 2x^3}{\sqrt{1 - x^4}}$.

(4 marks)

(b) If $M = \begin{bmatrix} 2 & p \\ -3p & 1 \end{bmatrix}$, find M^{-1}.

(2 marks)

QUESTION 2

Let $A = \begin{bmatrix} 1 & a \\ 0 & -1 \end{bmatrix}$.

(a) Find:

(i) A^2.

(2 marks)

(ii) A^3.

(1 mark)

(iii) A^4.

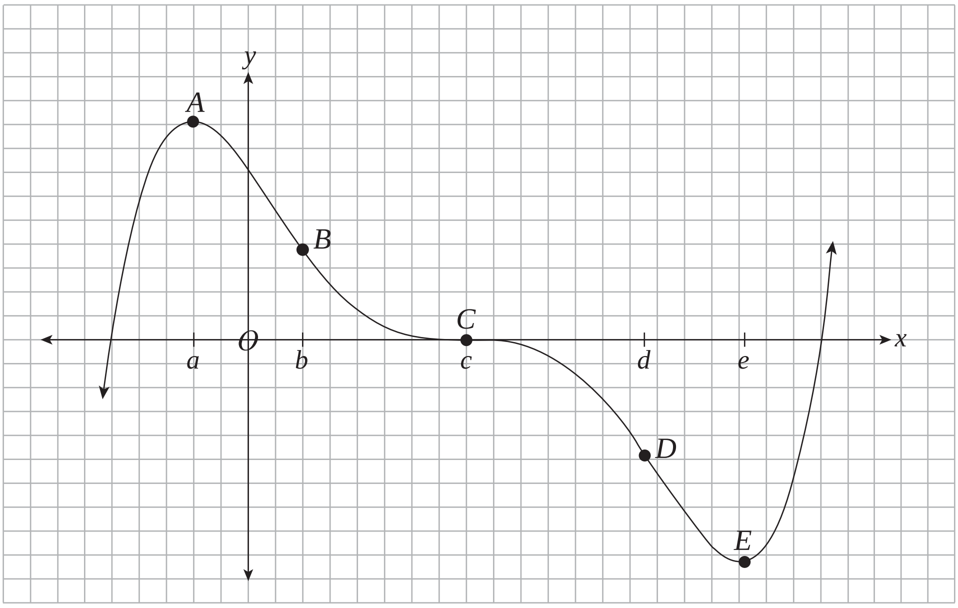
(1 mark)

(b) Using your results from part (a), write an expression for A^n, where n is a positive integer.

(2 marks)

QUESTION 3

Points A, B, C, D, and E, with x-coordinates a, b, c, d, and e, are the stationary points and the inflection points of the graph of $y = f(x)$, a fifth-order polynomial function, as shown below:



(a) (i) Which of points A, B, C, D, and E are the stationary points of the graph of $y = f(x)$?

(1 mark)

(ii) Which of points A, B, C, D, and E are the inflection points of the graph of $y = f(x)$?

(2 marks)

(b) Hence draw sign diagrams for $f'(x)$ and $f''(x)$.

$f'(x)$

x

$f''(x)$

x

(4 marks)

QUESTION 4

Consider the following system of linear equations:

$$\begin{aligned} a + 2b - c &= 4 \\ 3a - 4b + 2c &= 2 \\ 2a + 8b - 4c &= 1. \end{aligned}$$

Show, using clearly defined row operations, that this system of linear equations has no solution.

(4 marks)

QUESTION 5

During heavy rains in January 2009 the Bureau of Meteorology measured the height of the Herbert River.

The data from 48 hours of measurement have been analysed.

On the basis of these data, H, the height of the river in metres after t hours, can be modelled using the function

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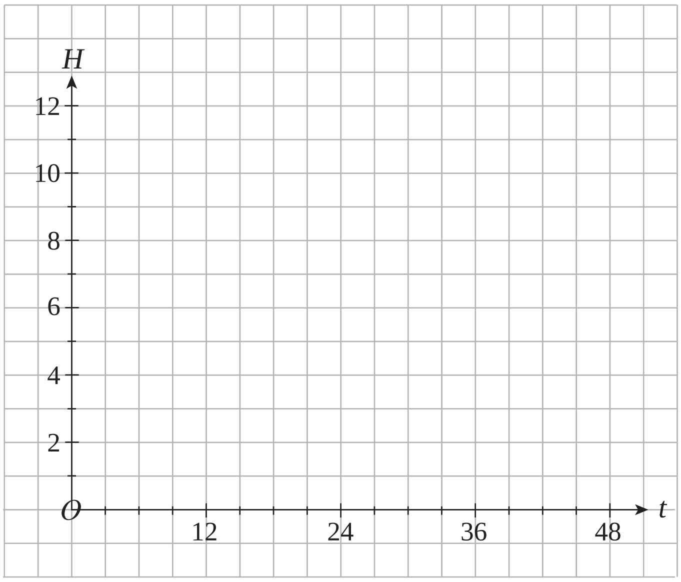
Source: http://media.photobucket.com

$$H = \left(2.29\times10^{-5}\right)t^4 - \left(2.36\times10^{-3}\right)t^3 + \left(7.25\times10^{-2}\right)t^2 - \left(5.38\times10^{-1}\right)t + 6.76.$$

(a) On the axes below, sketch the graph of H against t.



(2 marks)

(b) According to this model, what was the:

(i) maximum height of the river during this period?

(1 mark)

(ii) time when the height of the river was rising most quickly?

(2 marks)

(c) When the height of the river exceeds 10 metres there is a *moderate flood risk*.

(i) Add the horizontal line $H = 10$ to your graph.

(1 mark)

(ii) For how long, during the 48-hour period, was there a moderate flood risk, according to the model?

Give your answer to the nearest tenth of an hour.

(2 marks)

QUESTION 6

Consider the system of linear equations

$$\begin{aligned} x \;+ 2y + 5z &= 2 \\ 2x \;+ 2y + 3z &= -1 \\ (k+1)x \;+\; k^2 z &= 3, \end{aligned}$$

where k is a real number.

(a) This system of linear equations can be written as a matrix equation $AX = B$. Write down the matrices A, X, and B.

(2 marks)

(b) Show that $|A| = -2k^2 - 4k - 4$.

(3 marks)

(c) (i) Hence show that A has an inverse for all values of k.

(2 marks)

(ii) What is the significance of part (c)(i) in relation to the solution of the given system of linear equations?

(1 mark)

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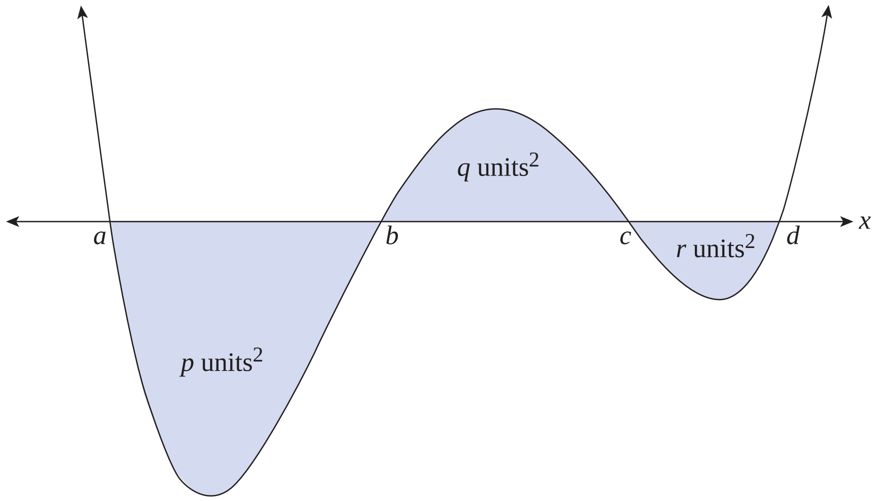
QUESTION 7

Consider the graph of $y = f(x)$. The area of each shaded region is shown on the graph below:



(a) Determine the value of each of the following definite integrals in terms of p, q, and r.

(i) $\int_b^c f(x)\,dx$.

(1 mark)

(ii) $\int_a^b f(x)\,dx$.

(1 mark)

(iii) $\int_a^d f(x)\,dx$.

(1 mark)

(b) Given that $p > q > r$, is $\int_a^d f(x)\,dx > 0$?

(1 mark)

You may write on this page if you need more space to finish your answers. Make sure to label each answer carefully (e.g. 'Question 6(c)(i) continued').

QUESTION 8

(a) A plant wholesaler is selling a large quantity of seeds. To determine the proportion of these seeds that will germinate and grow into seedlings, the wholesaler plants a random sample of 100 seeds and finds that eighty-seven germinate and grow into seedlings.

(i) Using this information, calculate a 95% confidence interval for the proportion of these seeds that will germinate and grow into seedlings.

(2 marks)

(ii) On the basis of this confidence interval, the wholesaler claims that at least 80% of the seeds will germinate and grow into seedlings.

Explain why it is reasonable for the wholesaler to make this claim.

(1 mark)

For the remainder of this question it is assumed that 80% of the seeds will germinate and grow into seedlings.

The manager of a plant nursery is planning to plant these seeds in trays consisting of six pots (as shown in the photograph).

Only trays with one or more seedlings in all six pots can be sold.

The manager wants to make sure that more than 90% of trays can be sold.

(b) The manager decides to plant two seeds in each pot.

(i) Calculate the probability that, if two seeds are planted in a single pot, at least one of the seeds will germinate and grow into a seedling.

(2 marks)

(ii) Using your answer to part (b)(i), calculate the probability that, if two seeds are planted in each of the six pots in a tray, at least one of the seeds in each pot will germinate and grow into a seedling.

(2 marks)

(c) If three seeds are planted in each of the six pots in a tray, is it likely that more than 90% of trays can be sold?

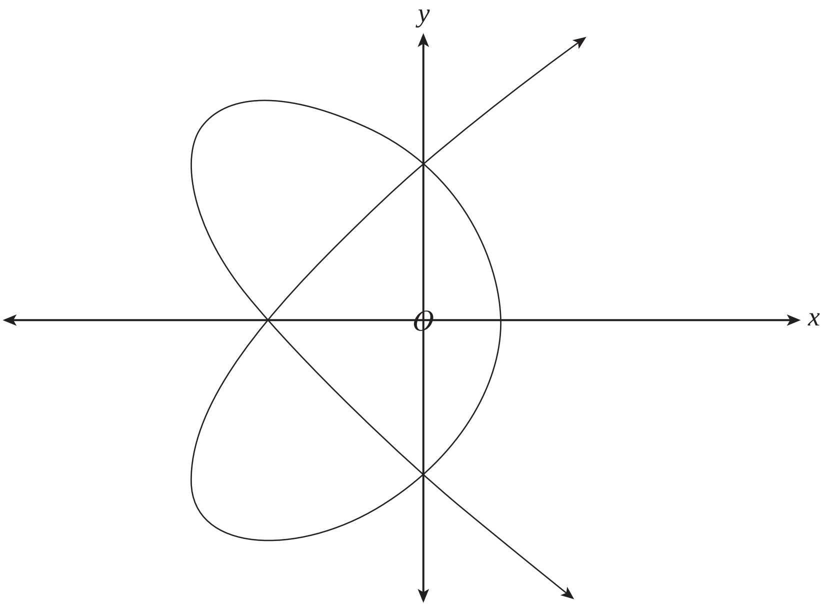
Show calculations to support your answer.

(4 marks)

PLEASE TURN OVER

QUESTION 9

The relation $2x^3+3x^2-\left(y^2-1\right)^2=0$ is graphed below:



(a) Find the x-coordinates of the points on the graph where $y=\pm\sqrt{2}$.

(2 marks)

(b) Show that $\dfrac{dy}{dx}=\dfrac{3x(x+1)}{2y\left(y^2-1\right)}$.

(3 marks)

(c) (i) On the graph opposite, mark with P the point(s) where the tangent to the graph of the relation is horizontal.

(1 mark)

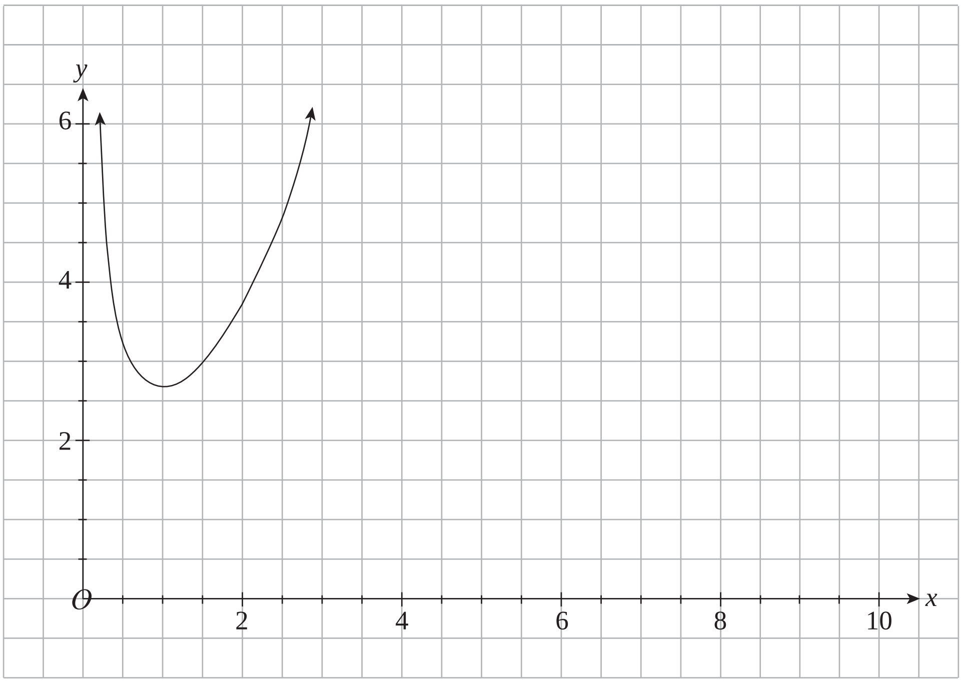
(ii) Using your results from parts (a) and (b), find the coordinates of the point(s) where the tangent to the graph of the relation is horizontal.

(3 marks)

PLEASE TURN OVER

QUESTION 10

The graph of $y = \frac{1}{x}e^x$ is shown below, for $x \geq 0$:



(a) Find the x-coordinate of the stationary point visible in the graph above.

(1 mark)

(b) On the axes above, sketch the graph of $y = \frac{1}{x^2}e^x$.

Accurately plot, and label with an M, the stationary point of this graph.

(2 marks)

(c) Complete the following table.

function	$y = \frac{1}{x}e^x$	$y = \frac{1}{x^2}e^x$	$y = \frac{1}{x^3}e^x$	$y = \frac{1}{x^4}e^x$
x-coordinate of stationary point				

(2 marks)

(d) Make a conjecture about the x-coordinate of the stationary point of $y = \frac{1}{x^n} e^x$, where n is a positive integer.

(1 mark)

(e) Prove or disprove the conjecture you made in part (d).

(3 marks)

QUESTION 11

The diagram (top right) shows an electricity cable suspended from two poles. The poles are 30 metres apart on level ground.

The cable is attached to Pole 1 at a point 11 metres above the ground, and it is attached to Pole 2 at a point 14 metres above the ground.

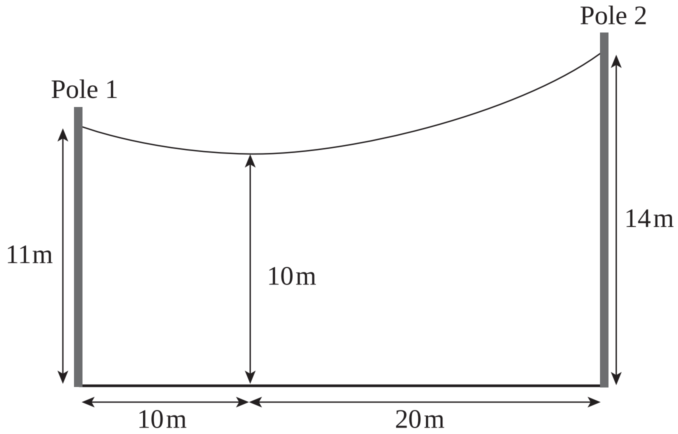
The lowest point of the cable is 10 metres from Pole 1 and 10 metres above the ground.

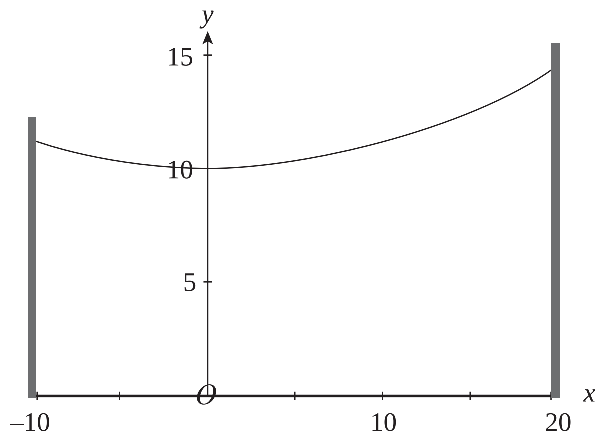
If ground level is represented by the x-axis and a vertical axis passes through the lowest point of the cable then the shape of the curve made by the cable can be modelled by the following function:

$$y = A + Be^{\frac{x}{100}} + Ce^{-\frac{x}{100}},$$

where A, B, and C are constants.

This model is shown in the diagram (bottom right).





(a) (i) Given that the lowest point of the cable is 10 metres above the ground, write a linear equation connecting A, B, and C.

(1 mark)

(ii) Using the information about the height of the cable at each pole, write two more linear equations connecting A, B, and C.

(2 marks)

(iii) Solve the system of three linear equations to find the values of A, B, and C, correct to four significant figures.

(2 marks)

(b) (i) Assuming that the model for the shape of the curve made by the cable is

$$y = -190 + 100e^{\frac{x}{100}} + 100e^{-\frac{x}{100}},$$

find $\frac{dy}{dx}$.

(2 marks)

(ii) The electricity company requires the slope of the cable, at the points where it is attached to the poles, to be between –0.5 and 0.5.

Show that this model satisfies this requirement at both poles.

(2 marks)

QUESTION 12

In extremely hot weather South Australia's electricity consumption is very high, mainly because of the increased use of air conditioners. As a result, there can be disruptions to the supply of electricity.

A device is invented with the aim of reducing household electricity use by changing the amount of electricity used by air conditioners.

To test the ability of the device to reduce household electricity use, a study is designed in which:

- a random sample of sixty-two houses in South Australia, each with similar air conditioners, are equipped with the device
- the sample is called Sample 1
- 2 days of similar weather conditions are chosen, Day 1 and Day 2
- on Day 1 the device is active
- on Day 2 the device is inactive
- the amount of household electricity used on each day is recorded.

Let:

C_A be the amount of household electricity used on the *active* day, measured in megajoules, to the nearest megajoule;

C_N be the amount of household electricity used on the *inactive* day, measured in megajoules, to the nearest megajoule;

X be $C_A - C_N$ for all houses in South Australia with similar air conditioners.

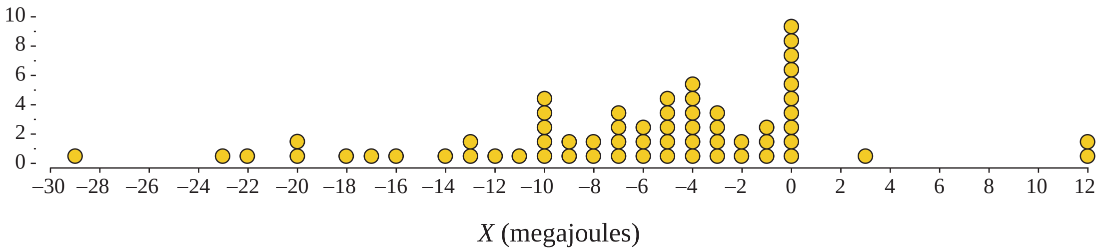
For one of the houses in Sample 1:

42 megajoules are used on the active day;

50 megajoules are used on the inactive day;

hence $x = -8$ megajoules.

A plot of X values for Sample 1 is shown below:



For Sample 1:

mean $\bar{x} = -6$ megajoules.

To test the hypotheses $\begin{array}{l} H_0 : \mu = 0 \\ H_A : \mu \neq 0 \end{array}$,

a two-tailed *Z*-test, at the 0.05 level of significance, is to be applied.

(a) Interpret the null hypothesis in terms of the study.

(2 marks)

(b) Assume that X can be modelled by a normal distribution with a standard deviation of $\sigma = 7$.

Sample 1 has a mean of $\bar{x} = -6$ megajoules.

Determine whether or not, on the basis of Sample 1, the null hypothesis should be rejected.

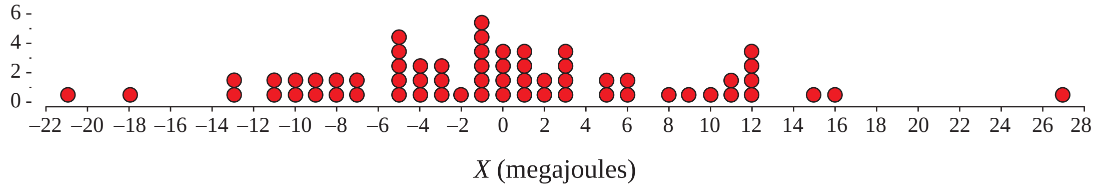
(3 marks)

(c) What can you conclude from your answer to part (b)?

(1 mark)

As part of the study a *different* random sample of sixty-two houses, each with similar air conditioners, are also equipped with the device. However, for this sample, the device is active on Day 2 and inactive on Day 1. This sample is called Sample 2.

A plot of X values for Sample 2 is shown below:



For Sample 2:

mean $\bar{x} = -1$ megajoules.

(d) Calculate a 95% confidence interval for μ, based on Sample 2.

Assume that X can be modelled by a normal distribution with a standard deviation of $\sigma = 7$.

(2 marks)

(e) Is the confidence interval you have calculated in part (d) consistent with your result in part (b)? Explain your answer.

(2 marks)

You may write on this page if you need more space to finish your answers. Make sure to label each answer carefully (e.g. 'Question 6(c)(i) continued').

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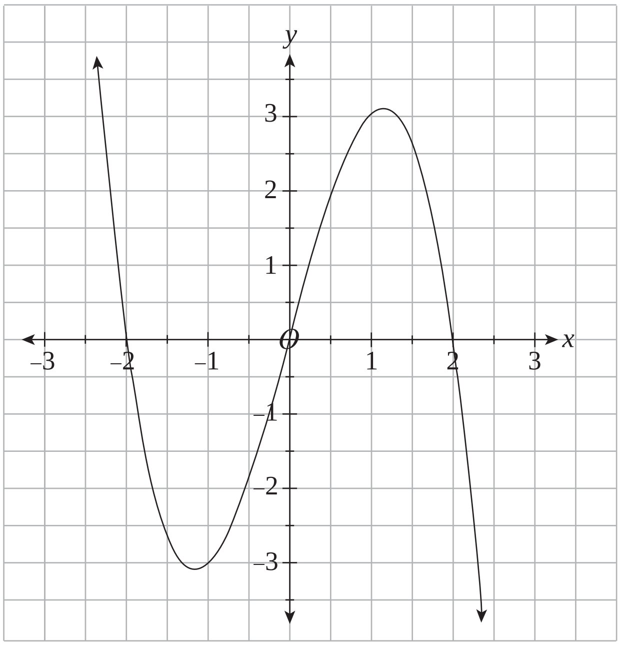
QUESTION 13

Consider the function $f(x) = -x^3 + 4x$.

The graph of $y = f(x)$ is shown below:



(a) Consider the linear function $g(x) = mx$, where m is a positive constant.

(i) On the graph above, sketch $g(x)$ for the case where $m = 2$. (1 mark)

(ii) Show that $f(x)$ and $g(x)$ intersect when $x = 0$ and $x = \pm\sqrt{4-m}$.

(3 marks)

(iii) For what values of m will $f(x)$ and $g(x)$ intersect at three points?

(1 mark)

(b) (i) Write an integral expression for the area enclosed by the graphs of $f(x)$ and $g(x)$ when they intersect at three points.

(3 marks)

(ii) Hence find the area enclosed by $f(x)$ and $g(x)$.

Simplify your answer to the form $k(4-m)^n$, where k and n are real numbers.

(4 marks)

QUESTION 14

The price, in dollars, of one share in a mineral exploration company varies, depending largely on the success of the company's explorations.

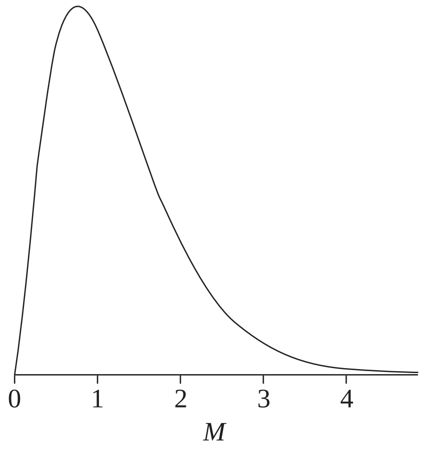
In relation to such a share, let $M = \dfrac{\text{the price of one share 1 year from now}}{\text{the price of one share now}}$.

(a) Explain what an M value of 1.5 implies about the change in the price of one share over 1 year.

(1 mark)

Consider the price of one share in each of a large number of mineral exploration companies. For these shares, let M be modelled by a non-normal distribution with mean $\mu_M = 1.25$ and standard deviation $\sigma_M = 3.2$. A table of probabilities for m, selected M values, is constructed. This table and the probability distribution of M are shown below:

m	0.5	1.0	1.25	1.5	2.0	2.5	3.0
$P(M < m)$	0.15	0.45	0.58	0.69	0.84	0.92	0.97



(b) Using the table above, find the probability that the price of one share in a randomly chosen mineral exploration company will:

(i) decrease over 1 year.

(1 mark)

(ii) increase by more than 50% over 1 year.

(1 mark)

Two investors, Bernard and Charles, each have the same amount of money to invest in mineral exploration companies:

- For his investment, Bernard decides to buy one share in each of 100 randomly chosen mineral exploration companies.
- For his investment, Charles decides to buy 100 shares in one randomly chosen mineral exploration company.

The price of all these shares is approximately the same at the time of purchase.

(c) Let $\overline{M}$ denote the average M value of Bernard's shares.

Assume that the M values of Bernard's shares vary independently.

(i) Find the mean $\mu_{\overline{M}}$ and the standard deviation $\sigma_{\overline{M}}$.

(2 marks)

(ii) Would it be reasonable to assume that the distribution of $\overline{M}$ is approximately normal? Explain your answer.

(2 marks)

(iii) Find the probability that the price of Bernard's investment will decrease over 1 year.

(1 mark)

PLEASE TURN OVER

(iv) Find the probability that the price of Bernard's investment will increase by more than 50% over 1 year.

(1 mark)

(d) On the basis of your answers to parts (b) and (c), describe the difference between Bernard's investment decision and Charles's investment decision.

(2 marks)

You may write on this page if you need more space to finish your answers. Make sure to label each answer carefully (e.g. 'Question 6(c)(i) continued').

QUESTION 15

In a certain metal-smelting factory, metal objects are cast and then polished. After polishing, the objects are inspected and classified as one of the following:

- Finished — ready for sale
- Unfinished — needing further polishing
- Rejected — not fit for sale and melted down.

At all inspections 70% of objects are classified as Finished, 20% are classified as Unfinished, and 10% are Rejected.

If an object is Unfinished, it is returned for further polishing and then is inspected and classified again.

(a) If 1000 objects are cast, calculate how many will be Finished after, at most, two polishes.

(2 marks)

(b) Let $M = \begin{bmatrix} 0 & 1000 & 0 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 0 & 0 \\ 0.7 & 0.2 & 0.1 \\ 0 & 0 & 1 \end{bmatrix}$.

(i) Calculate MP.

(1 mark)

(ii) Calculate MP^2.

(1 mark)

(iii) Interpret your result in part (b)(ii).

(1 mark)

(c) Write down the matrix product that determines the classification of objects after, at most, five polishes.

(1 mark)

(d) By considering the matrix P^n for large n, find the proportion of objects that will eventually be Rejected if there is no limit to the number of times that an object can be returned for further polishing.

(2 marks)

(e) The process is changed so that objects that are Unfinished after two polishes will be Rejected, and no further polishing will take place.

(i) Write down a matrix Q such that the matrix product MPQ determines the classification of objects, using the changed process.

(2 marks)

(ii) Hence, or otherwise, determine how many objects will be classified as Finished, as Unfinished, and as Rejected if 1000 objects are cast and polished, using the changed process.

(1 mark)

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QUESTION 16

Car A (shown in the photograph) competes in ANDRA Pro Series drag races.

Source: www.autoclub.com.au

In such a race, cars are initially stationary on the starting line and then travel 400 metres in a straight line in as short a time as possible.

The design of Car A is considered in the development of a mathematical model to predict the speed (v_A, metres per second) of the car t seconds after it leaves the starting line.

The model assumes that all the components of Car A work optimally and that the driver attempts to record as short a time as possible. The model is useful for predictions only for the first 7 seconds after the car leaves the starting line.

The model is shown below:

$$v_A = \frac{98}{1+19e^{-2t}} - 4.9 \quad \text{for } 0 \le t \le 7.$$

(a) Find the maximum speed of Car A, according to this model.

(1 mark)

(b) (i) Calculate $\int_0^2 v_A \, dt$, correct to the nearest whole number.

(1 mark)

(ii) What does your answer to part (b)(i) mean in terms of Car A in a drag race?

(1 mark)

(c) Complete the following table, according to the model on page 32.

t (seconds)	0	2	4	6
distance (metres) travelled by Car A after *t* seconds	0			

(1 mark)

The photograph shows Car B, a drag racing car under construction.

The current design, and the assumptions that apply to Car A, are considered in the development of a mathematical model to predict the speed (v_B, metres per second) of Car B *t* seconds after it leaves the starting line.

The model is shown below:

This photograph cannot be reproduced here for copyright reasons.

Source: www.pro-touring.com

$$v_B = \frac{85}{1 + 9e^{-3t}} - 8.5 \quad \text{for } 0 \le t \le 7.$$

(d) Complete the following table, according to the model above.

t (seconds)	0	2	4	6
distance (metres) travelled by Car B after *t* seconds	0			

(2 marks)

(e) If Car A raced against Car B, which car do the models predict will win the race? Explain your answer.

(2 marks)

(f) (i) Find $\frac{\mathrm{d}y}{\mathrm{d}t}$ if $y = \frac{1}{2}\ln\left(e^{2t} + k\right)$, where k is a positive real constant.

(2 marks)

(ii) Hence show that $\int \frac{98}{1+19e^{-2t}} - 4.9\,\mathrm{d}t = 49\ln\left(e^{2t} + 19\right) - 4.9t + c.$

(3 marks)

(iii) Hence determine the time Car A takes to travel the 400 metres.

Give your answer correct to two decimal places.

(3 marks)

You may write on this page if you need more space to finish your answers. Make sure to label each answer carefully (e.g. 'Question 6(c)(i) continued').

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You may remove this page from the booklet by tearing along the perforations so that you can refer to it while you write your answers.

LIST OF MATHEMATICAL FORMULAE FOR USE IN STAGE 2 MATHEMATICAL STUDIES

Standardised Normal Distribution

A measurement scale X is transformed into a standard scale Z, using the formula

$$Z = \frac{X-\mu}{\sigma}$$

where μ is the population mean and σ is the standard deviation for the population distribution.

Confidence Interval — Mean

A 95% confidence interval for the mean μ of a normal population with standard deviation σ, based on a simple random sample of size n with sample mean $\bar{x}$, is

$$\bar{x} - 1.96\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{x} + 1.96\frac{\sigma}{\sqrt{n}}.$$

For suitably large samples, an approximate 95% confidence interval can be obtained by using the sample standard deviation s in place of σ.

Sample Size — Mean

The sample size n required to obtain a 95% confidence interval of width w for the mean of a normal population with standard deviation σ is

$$n = \left(\frac{2 \times 1.96\sigma}{w}\right)^2.$$

Confidence Interval — Population Proportion

An approximate 95% confidence interval for the population proportion p, based on a large simple random sample of size n with sample proportion

$\hat{p} = \frac{X}{n}$, is

$$\hat{p} - 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le p \le \hat{p} + 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}.$$

Sample Size — Proportion

The sample size n required to obtain an approximate 95% confidence interval of approximate width w for a proportion is

$$n = \left(\frac{2 \times 1.96}{w}\right)^2 p^*\left(1-p^*\right).$$

(p^* is a given preliminary value for the proportion.)

Binomial Probability

$$P(X=k) = C_k^n p^k (1-p)^{n-k}$$

where p is the probability of a success in one trial and the possible values of X are $k = 0, 1, \ldots n$ and

$$C_k^n = \frac{n!}{(n-k)!k!} = \frac{n(n-1)\ldots(n-k+1)}{k!}.$$

Binomial Mean and Standard Deviation

The mean and standard deviation of a binomial count X and a proportion of successes $\hat{p} = \frac{X}{n}$ are

$$\mu_X = np \qquad \mu(\hat{p}) = p$$

$$\sigma_X = \sqrt{np(1-p)} \qquad \sigma(\hat{p}) = \sqrt{\frac{p(1-p)}{n}}$$

where p is the probability of a success in one trial.

Matrices and Determinants

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\det A = |A| = ad - bc$ and

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Derivatives

$f(x) = y$	$f'(x) = \frac{dy}{dx}$
x^n	nx^{n-1}
e^{kx}	ke^{kx}
$\ln x = \log_e x$	$\frac{1}{x}$

Properties of Derivatives

$$\frac{d}{dx}\{f(x)g(x)\} = f'(x)g(x) + f(x)g'(x)$$

$$\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

Quadratic Equations

If $ax^2 + bx + c = 0$ then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$