Government of South Australia

SACE Board of SA

External Examination 2008

2008 SPECIALIST MATHEMATICS

FOR OFFICE USE ONLY

SUPERVISOR CHECK

RE-MARKED

ATTACH SACE REGISTRATION NUMBER LABEL TO THIS BOX

Graphics calculator ☐

Brand ____________

Model ____________

Computer software ☐

Friday 14 November: 9 a.m.

Time: 3 hours

Pages: 45
Questions: 16

Examination material: one 45-page question booklet
one SACE registration number label

Approved dictionaries, notes, calculators, and computer software may be used.

Instructions to Students

1. You will have 10 minutes to read the paper. You must not write in your question booklet or use a calculator during this reading time but you may make notes on the scribbling paper provided.

2. This paper consists of three sections:

Section A (Questions 1 to 10) Answer ***all*** questions in Section A.	75 marks
Section B (Questions 11 to 14) Answer ***all*** questions in Section B.	60 marks
Section C (Questions 15 and 16) Answer ***one*** question from Section C.	15 marks

3. Write your answers in the spaces provided in this question booklet. There is no need to fill all the space provided. You may write on pages 23, 34, and 44 if you need more space, making sure to label each answer clearly.

4. Appropriate steps of logic and correct answers are required for full marks.

5. Show all working in this booklet. (You are strongly advised ***not*** to use scribbling paper. Work that you consider incorrect should be crossed out with a single line.)

6. Use only black or blue pens for all work other than graphs and diagrams, for which you may use a sharp dark pencil.

7. State all answers correct to three significant figures, unless otherwise stated or as appropriate.

8. Diagrams, where given, are not necessarily drawn to scale.

9. The list of mathematical formulae is on page 45. You may remove the page from this booklet before the examination begins.

10. Complete the box on the top right-hand side of this page with information about the electronic technology you are using in this examination.

11. Attach your SACE registration number label to the box at the top of this page.

SECTION A (Questions 1 to 10)

(75 marks)

*Answer **all** questions in this section.*

QUESTION 1 (6 marks)

A triangle has vertices $A(1, -2, 3)$, $B(-3, 2, 5)$, and $C(4, 7, 1)$.

(a) (i) Find $\overrightarrow{AB}$.

(1 mark)

(ii) Find $|\overrightarrow{AB}|$.

(1 mark)

(b) Find $\angle BAC$.

(4 marks)

QUESTION 2 (5 marks)

$P(x) = x^3 - 2x^2 + ax + b$, where a and b are real constants.

(a) When $P(x)$ is divided by $x - 1$, the remainder is 6.

Show that $a + b = 7$.

(2 marks)

(b) If $P(x)$ also has a factor of $x + 2$, find a and b.

(3 marks)

QUESTION 3 (6 marks)

Figure 1 shows triangle ABC with altitudes BP and CQ. Point O is the midpoint of BP. The circle with centre O and radius OP cuts AB at point R. Line segment OQ is perpendicular to BP.

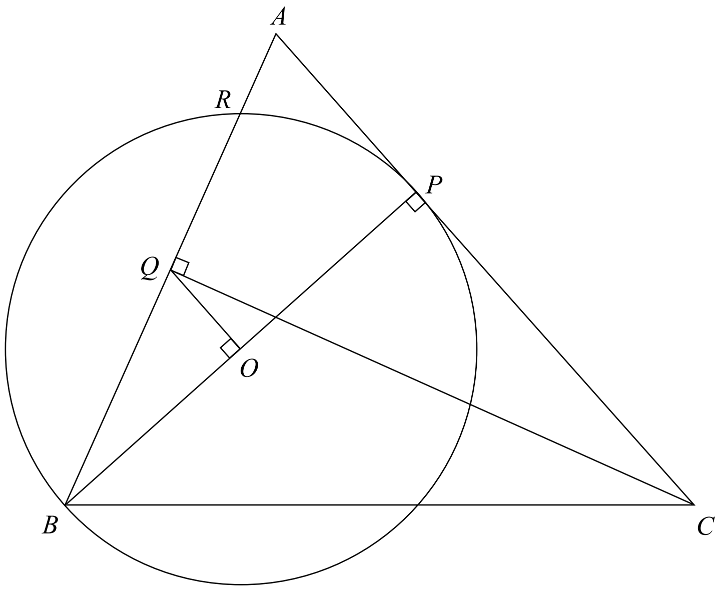

Figure 1

(a) On Figure 1 draw line segment PR.

Prove that PR is parallel to CQ.

(2 marks)

(b) Explain why $\angle RPA = \angle RBP$.

(1 mark)

(c) Prove that quadrilateral $OPRQ$ is cyclic.

(1 mark)

(d) Hence, or otherwise, prove that $\angle ORQ = \angle RPA$.

(2 marks)

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QUESTION 4 (8 marks)

Let $v = 6 + 8i$ and $w = 7 + i$.

(a) On the Argand diagram in Figure 2, plot the points corresponding to v and w and label them V and W respectively.

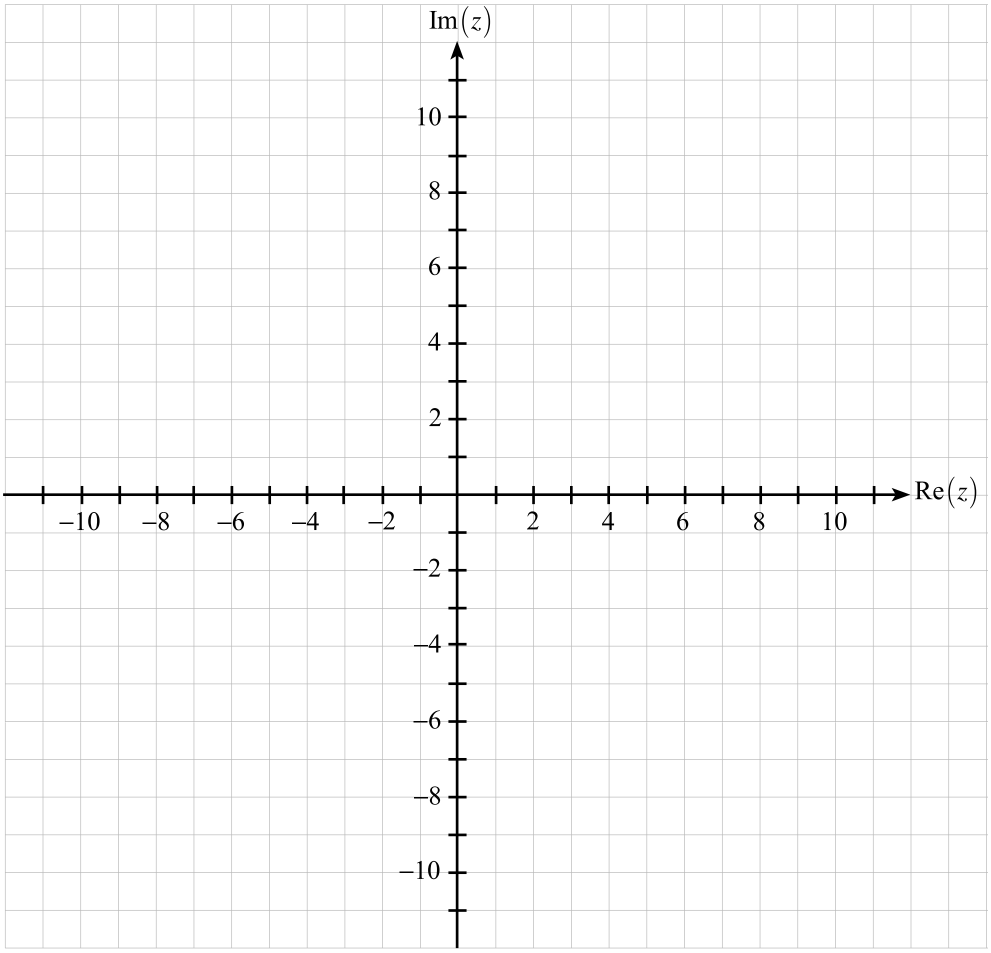

Figure 2 (1 mark)

(b) On the Argand diagram in Figure 2, draw the set of complex numbers z such that $|z| = 10$.

(2 marks)

(c) Verify that v is a member of the set of complex numbers drawn in part (b).

(1 mark)

(d) Let u be a complex number such that $u + i\overline{w} = \overline{w}$.

Find u in Cartesian form.

(2 marks)

(e) On the Argand diagram in Figure 2, mark the set of complex numbers z that satisfy both $|z| = 10$ and $|z - u| = |z - v|$.

(2 marks)

QUESTION 5 (6 marks)

(a) Perform four iterations of the transformation $z \longrightarrow z^2 + i$, given that $z_0 = i$.

(3 marks)

(b) Calculate the modulus of each result found in part (a) and describe the behaviour of the iteration.

(3 marks)

QUESTION 6 (7 marks)

A pharmaceutical company markets an antibiotic tablet that has the shape of a cylinder with hemispherical ends, as shown in Figure 3. The surface area of the tablet is 200 square millimetres. The cylindrical section has a length of l millimetres and a radius of r millimetres.

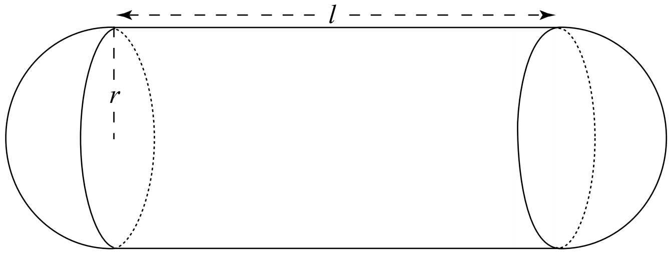

Figure 3

(a) (i) Show that the surface area of the tablet is $A = 2\pi rl + 4\pi r^2$.

(1 mark)

(ii) Hence show that $\frac{dA}{dt} = (2\pi l + 8\pi r)\frac{dr}{dt} + 2\pi r\frac{dl}{dt}$.

(3 marks)

(b) To answer this part, use the following formulae from part (a):

$$A = 2\pi rl + 4\pi r^2$$

$$\frac{dA}{dt} = (2\pi l + 8\pi r)\frac{dr}{dt} + 2\pi r\frac{dl}{dt}.$$

At a particular instant when the tablet is dissolving:

- the radius is 1 millimetre and is decreasing at the rate of 0.05 millimetres per second;
- the surface area is half its original value and is decreasing at the rate of 6 square millimetres per second.

Find the rate at which the length is changing at this instant.

(3 marks)

QUESTION 7 (11 marks)

(a) (i) By solving the differential equation $\frac{dy}{dx} = 3y - 6xy$, with the initial condition $x = 0,\ y = 2$, show that $y = 2e^{3x(1-x)}$.

(4 marks)

(ii) On the axes in Figure 4, sketch the solution showing the initial condition.

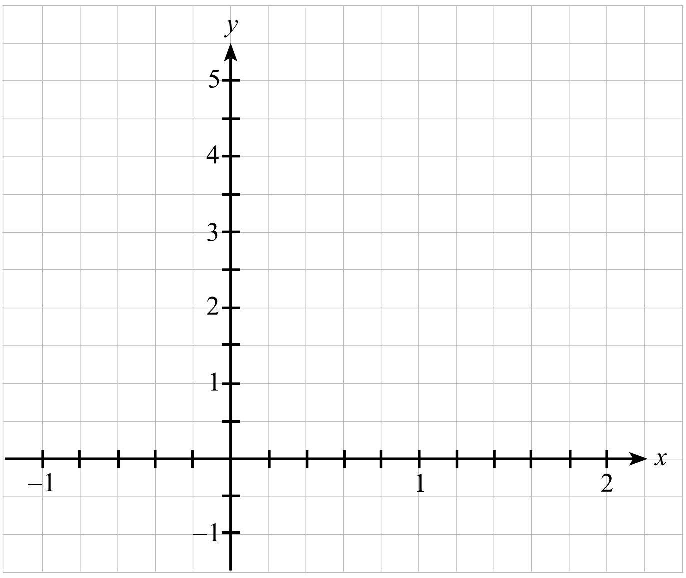

Figure 4

(3 marks)

Source: Pacifica Ocean Discovery Center,
http://oceandiscoverycenter.org/images/earthday/2005/2005-Pages/Image4.html

(b) A bike mound in a park is to be made with a uniform cross-sectional shape described by the graph of $y = 2e^{3x(1-x)}$ and the ground level at $y = 1$.

(i) Write an expression to find the cross-sectional area for a bike mound of this shape.

(2 marks)

(ii) Hence find an approximate value for the cross-sectional area in part (b)(i).

(1 mark)

(iii) Find the approximate volume of dirt required to make the bike mound if it is to have a width of 2.5 units.

(1 mark)

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QUESTION 8 (8 marks)

(a) Show that:

(i) $(1+\text{cis}\,\theta)(1-\text{cis}\,\theta)=1-\text{cis}\,2\theta$.

(2 marks)

(ii) $1+\text{cis}\,\theta+\text{cis}\,2\theta=\dfrac{1-\text{cis}\,3\theta}{1-\text{cis}\,\theta},\ \text{cis}\,\theta\neq 1$.

(2 marks)

(b) Use an inductive argument to show that

$$1+\operatorname{cis}\theta+\operatorname{cis}2\theta+\ \ldots\ +\operatorname{cis}n\theta=\frac{1-\operatorname{cis}(n+1)\theta}{1-\operatorname{cis}\theta}$$ where $\operatorname{cis}\theta\neq 1$ and $n\geq 1$ is an integer.

(3 marks)

(c) Hence evaluate

$$1+\operatorname{cis}\frac{\pi}{180}+\operatorname{cis}\frac{2\pi}{180}+\ \ldots\ +\operatorname{cis}\frac{359\pi}{180}.$$

(1 mark)

QUESTION 9 (8 marks)

(a) (i) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ for $y = x\cos 2x$.

(2 marks)

(ii) Hence show that $\int x\sin 2x \ \mathrm{d}x = \frac{1}{4}\sin 2x - \frac{1}{2}x\cos 2x + c$, where c is a constant.

(2 marks)

(b) Figure 5 shows the graph of $y = x \sin 2x$ from $x = 0$ to $x = \pi$.

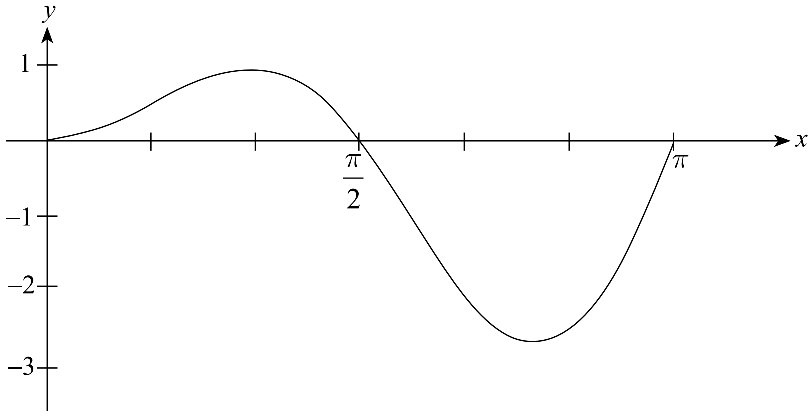


Figure 5

Referring to Figure 5, prove that one of the areas bounded by the curve $y = x \sin 2x$ and the x-axis is exactly three times the other.

(4 marks)

QUESTION 10 (10 marks)

Figure 6 shows the slope field for the differential system $\begin{cases} x' = 2x + 4y \\ y' = x - y \end{cases}$ and the point $P(2, -1)$.

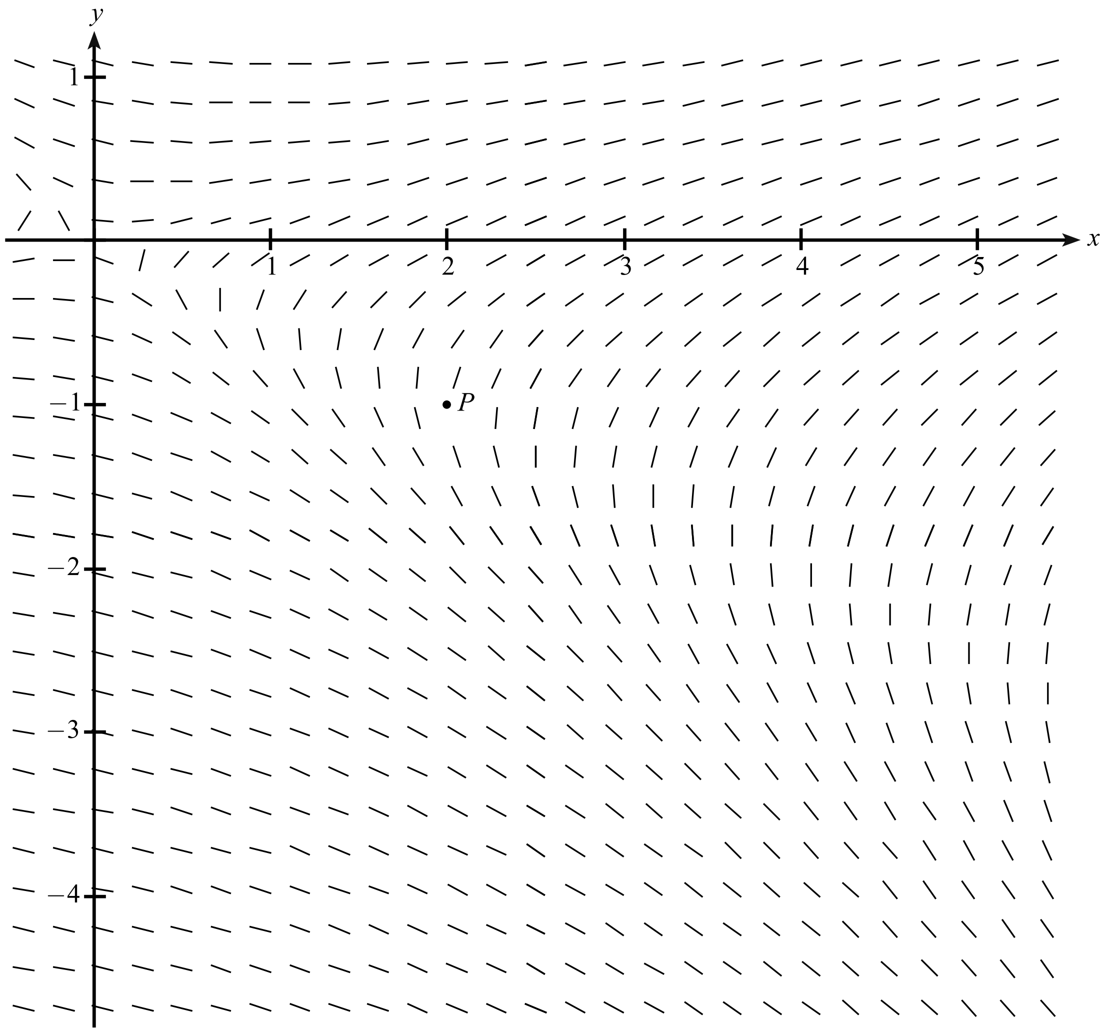


Figure 6

(a) On Figure 6 draw the solution curve that passes through P. (3 marks)

(b) The differential system has a general solution of the form $\begin{cases} x(t) = Ae^{3t} + Be^{-2t} \\ y(t) = Ce^{3t} + De^{-2t}. \end{cases}$

(i) Differentiate this solution to find an expression for $x'(t)$.

(1 mark)

(ii) Using the initial conditions $x = 2$, $y = -1$, show that $\begin{cases} A + B = 2 \\ 3A - 2B = 0. \end{cases}$

(2 marks)

(iii) Hence, or otherwise, find the particular solution that passes through P.

(4 marks)

PLEASE TURN OVER

SECTION B (Questions 11 to 14)

(60 marks)

*Answer **all** questions in this section.*

QUESTION 11 (15 marks)

(a) Given the points $A(2, 2, 0)$, $B(3, 4, 1)$, $C(4, 3, -4)$, and $D(8, 12, 2)$:

(i) find $\overrightarrow{AB} \times \overrightarrow{AC}$.

(2 marks)

(ii) find the equation of P_1, the plane through A, B, and C.

(2 marks)

(iii) show that D lies on plane P_1.

(1 mark)

(b) Consider the points $E(5,\ 3,\ -1)$ and $F(3,\ -3,\ -3)$.

(i) Show that E is closer than F to plane P_1.

(2 marks)

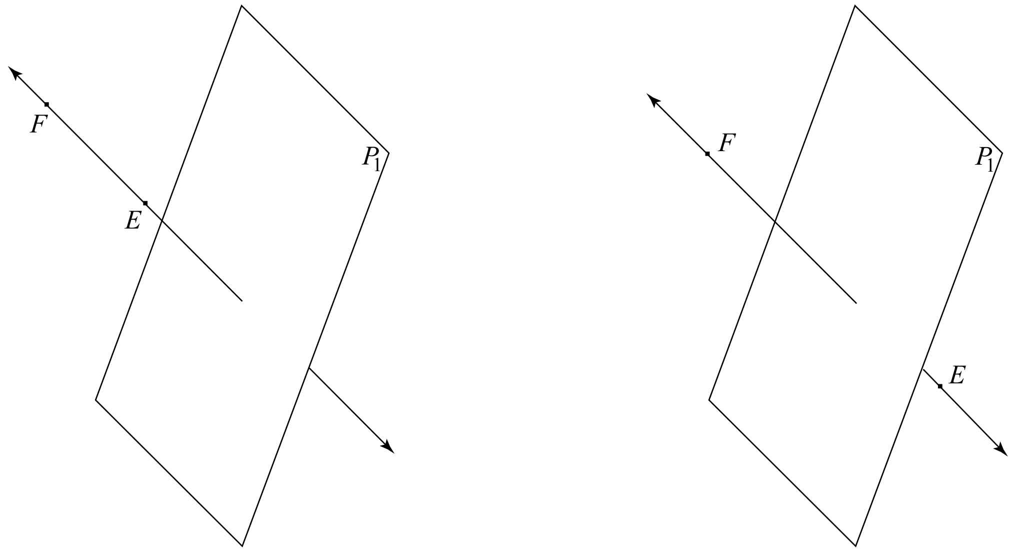
(ii) Find the equations of the line through E and F in parametric form.

(2 marks)

(iii) Show that the line in part (b)(ii) intersects plane P_1 at D.

(2 marks)

(c) Carefully explain whether E and F are on the same side or on opposite sides of plane P_1. See Figure 7.



E and F are on the same side of plane P_1 | E and F are on opposite sides of plane P_1

Figure 7

(4 marks)

You may write on this page if you need more space to finish your answers. Make sure to label each answer carefully (e.g. 'Question 8(a)(ii) continued').

QUESTION 12 (14 marks)

(a) Consider the function $f(x) = \dfrac{20 + 4\cos 2x}{7 - \cos 2x}$.

On the axes in Figure 8, sketch the curve of $y = f(x)$ for $0 \le x \le 2\pi$.

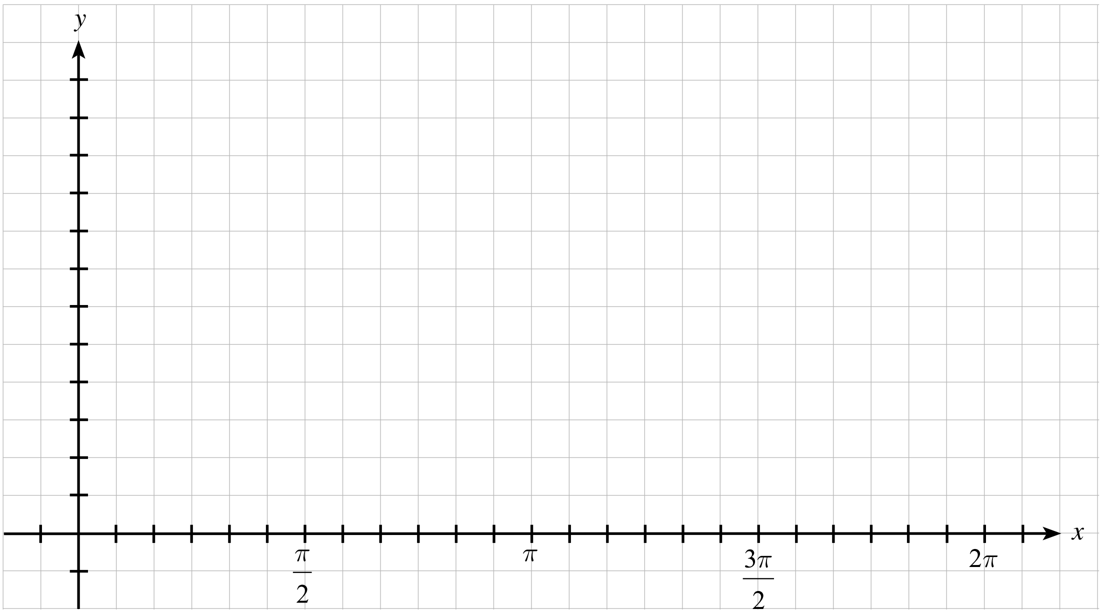


Figure 8

(3 marks)

(b) A formula used in the production of corrugated iron roofing sheets is given by

$$g(x) = \frac{a + b\cos 2x}{c - \cos 2x}$$

where a, b, and c are positive constants and $c > 1$.

Corrugated iron

Source: www.midcoasttimber.com.au/products/metal_roofing

(i) Show that $g(x)$ can be written in the form

$$g(x) = \frac{a + bc}{c - \cos 2x} - b.$$

(1 mark)

(ii) Hence, or otherwise, show that $g'(x)=\dfrac{-2(a+bc)\sin 2x}{(c-\cos 2x)^2}$.

(1 mark)

(iii) Explain why the maximum and minimum values of $g(x)$ are given by

$$g_{\max}=\frac{a+b}{c-1} \quad \text{and} \quad g_{\min}=\frac{a-b}{c+1}.$$

(2 marks)

(iv) Because of the way it is adjusted, the machine that produces the corrugated iron roofing sheets restricts the values of a, b, and c so that $a+b+c=37$ and the maximum and minimum values of $g(x)$ are 8 units and 2 units respectively.

Show that the restrictions mean that the values of a, b, and c must satisfy the system of equations

$a+b+c=37$ (given: restriction caused by machine)

$a+b-8c=-8$

$a-b-2c=2.$

(2 marks)

(v) State the values of a, b, and c that satisfy the system of equations given in part (b)(iv).

(1 mark)

(c) A new machine is installed that allows for less restricted values of a, b, and c, and the equation $a+b+c=37$ given in part (b)(iv) no longer applies.

(i) Show that the expression for the vertical height h between the maximum and minimum values of $g(x)$ is given by

$$h=\frac{2(a+bc)}{c^2-1} \text{ where } c>1.$$

(2 marks)

(ii) If the values for a and c are set at $a=22$ and $c=5$, find the rate of change of h with respect to b and interpret the result.

(2 marks)

QUESTION 13 (15 marks)

An ice skater moves in a path with her position defined by

$$\begin{cases} x(t) = 20\sin\left(\frac{t}{8}\right) \\ y(t) = -12\cos\left(\frac{t}{8}\right) \end{cases}$$

where x and y are in metres and $t \geq 0$ is in seconds.

(a) On the axes in Figure 9, sketch the path of the ice skater.

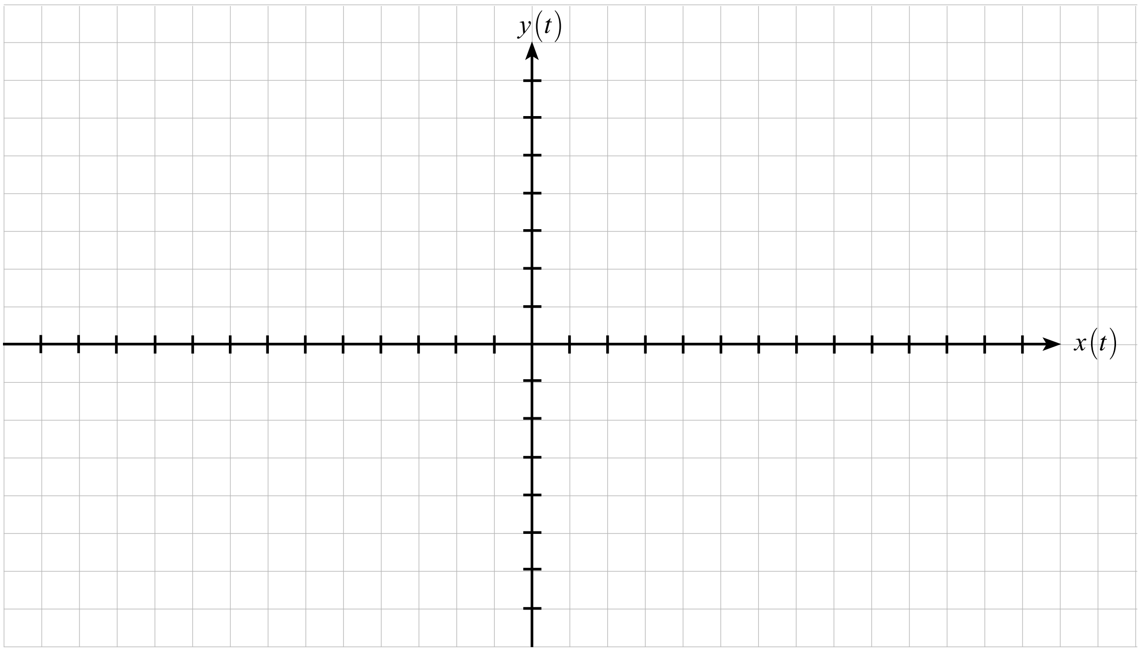


Figure 9 (3 marks)

(b) How long does it take for the ice skater to complete one circuit?

(1 mark)

(c) Find the velocity vector $\boldsymbol{v}(t)$ of the ice skater at time t seconds and hence show that the speed s of the ice skater is

$$s=\sqrt{6.25\cos^2\left(\frac{t}{8}\right)+2.25\sin^2\left(\frac{t}{8}\right)}.$$

(3 marks)

(d) Show that $\frac{\mathrm{d}}{\mathrm{d}t}\left(s^2\right)=k\cos\left(\frac{t}{8}\right)\sin\left(\frac{t}{8}\right)$, where k is constant.

(3 marks)

(e) Hence find the exact time(s) and position(s) when the ice skater is moving at the fastest speed during the first circuit.

(3 marks)

(f) As the ice skater moves along the path, she slips at $t = 6$ seconds.

(i) Find the ice skater's velocity vector at this time.

(1 mark)

(ii) Find the ice skater's position at this time.

(1 mark)

QUESTION 14 (16 marks)

(a) Using De Moivre's theorem, solve $z^4 = 81i$ for z, where z is a complex number.

(4 marks)

(b) Let the solutions of $z^4 = 81i$ be z_1, z_2, z_3, and z_4, with z_1 in quadrant 1 and arguments increasing anticlockwise from the positive $\text{Re}(z)$ axis.

On the Argand diagram in Figure 10, draw and label z_1, z_2, z_3, and z_4.

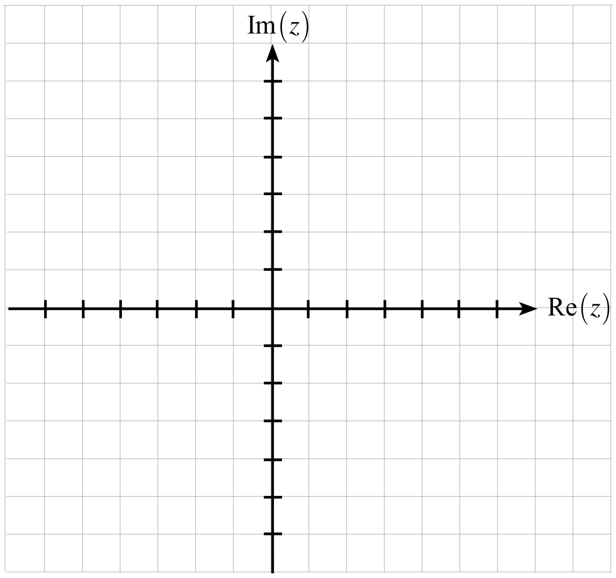


Figure 10

(2 marks)

(c) (i) On the Argand diagram in Figure 10, draw $z_1 - z_4$ and $z_2 - z_1$. (1 mark)

(ii) Show that $\left|\dfrac{z_1 - z_4}{z_2 - z_1}\right| = 1$ and $\arg\left(\dfrac{z_1 - z_4}{z_2 - z_1}\right) = -\dfrac{\pi}{2}$.

(4 marks)

(iii) Hence write $\dfrac{z_1 - z_4}{z_2 - z_1}$ in polar form.

(1 mark)

(d) Figure 11 represents the triangle in the Argand diagram in Figure 10 with vertices z_1, z_2, and z_4.

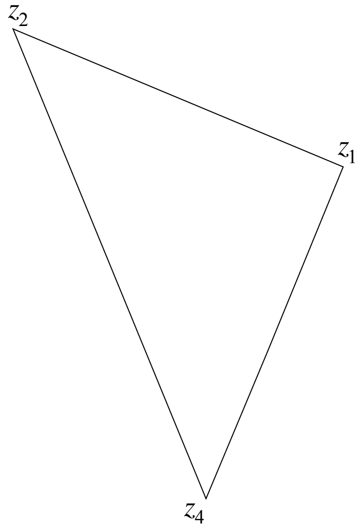


Figure 11

(i) By referring to Figure 11, explain why $|z_1 - z_4| + |z_2 - z_1| > |z_2 - z_4|$.

(2 marks)

(ii) Hence show that $\left|\dfrac{z_2 - z_4}{z_2 - z_1}\right| < 2$.

(2 marks)

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You may write on this page if you need more space to finish your answers. Make sure to label each answer carefully (e.g. 'Question 8(a)(ii) continued').

SECTION C (Questions 15 and 16)

(15 marks)

*Answer **one** question from this section, **either** Question 15 **or** Question 16.*

Answer ***either*** *Question 15* ***or*** *Question 16.*

QUESTION 15 (15 marks)

Consider the quadratic iteration $z \longrightarrow z^2 + c,\ z_0 = 0$, where c is a point in the Mandelbrot set.

(a) Complete the table of iterates below with $c = -1.755$.

n	z_n
0	0
1	
2	
3	
4	
5	
6	
7	
8	
9	

(3 marks)

(b) For the quadratic iteration $z \longrightarrow z^2 + c,\ z_0 = 0$, find z_1, z_2, and z_3 in terms of c.

(2 marks)

(c) Hence show that the three-cycle points in the Mandelbrot set are the zeros of the cubic polynomial $P(c) = c^3 + 2c^2 + c + 1$.

(2 marks)

(d) Without finding the zeros, explain why $P(c)$ must have at least one real zero, α, where $\alpha \neq 0$.

(1 mark)

(e) Let $P(c) = (c - \alpha)\left(c^2 + \beta c - \frac{1}{\alpha}\right)$.

Show that $\beta = 2 + \alpha$.

(2 marks)

(f) Graphically, or otherwise, find a value for α. Give your answer correct to four significant figures.

(1 mark)

(g) Using your answers from parts (c), (e), and (f), find all three zeros of $P(c)$ and plot them on the Argand diagram in Figure 12.

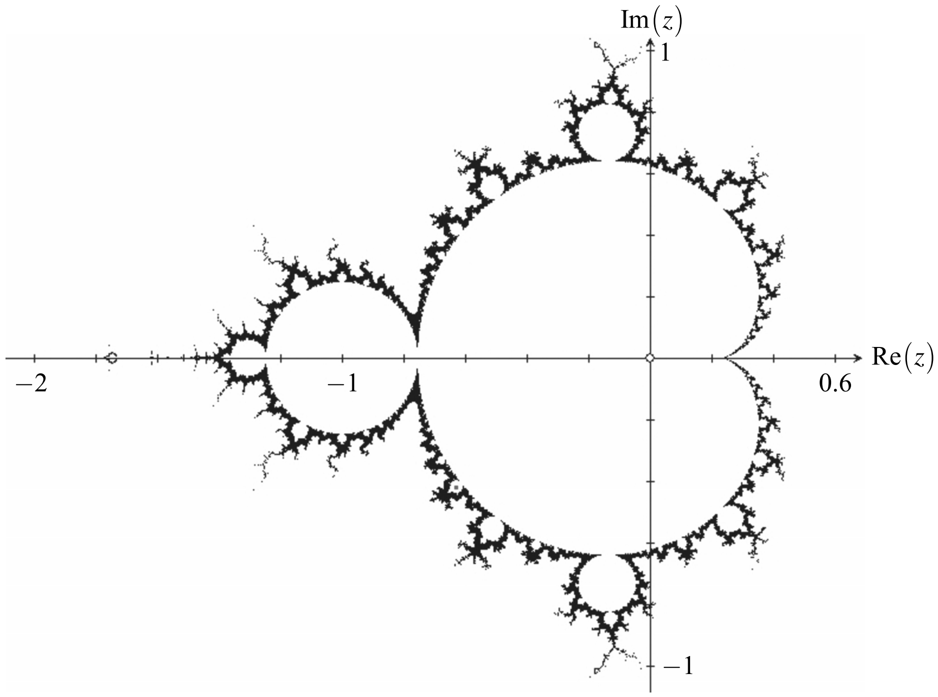

Figure 12 (3 marks)

(h) Give a value for $c \neq \alpha$ for which the quadratic iteration $z \longrightarrow z^2 + c$, $z_0 = 0$, is a three-cycle.

(1 mark)

*Answer **either** Question 15 **or** Question 16.*

QUESTION 16 (15 marks)

For a Bézier curve with start point $A(x_0, y_0)$, first control point $B(x_1, y_1)$, second control point $C(x_2, y_2)$, and endpoint $D(x_3, y_3)$, the general parametric equations are

$$\begin{cases} x(t) = (x_3 - 3x_2 + 3x_1 - x_0)t^3 + (3x_2 - 6x_1 + 3x_0)t^2 + (3x_1 - 3x_0)t + x_0 \\ y(t) = (y_3 - 3y_2 + 3y_1 - y_0)t^3 + (3y_2 - 6y_1 + 3y_0)t^2 + (3y_1 - 3y_0)t + y_0 \end{cases} \quad \text{where } 0 \le t \le 1.$$

A designer is producing a commercial logo using Bézier curves. One curve in the partially completed design uses the following points: start point $(-4, -2)$, first control point $(3, 5)$, second control point $(-3, 5)$, endpoint $(4, -2)$.

(a) The equation for $x(t)$ is $x(t) = 26t^3 - 39t^2 + 21t - 4$.

Find the equation for $y(t)$.

(1 mark)

(b) The next stage of the design involves reflecting the curve described above in the line $y = -2$.

(i) State the start point, first control point, second control point, and endpoint for the reflected curve.

(2 marks)

(ii) Using the new sets of coordinates, find the parametric equations for the reflected curve.

(2 marks)

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(c) On the axes in Figure 13, draw the curves from parts (a) and (b)(ii).

Show the start point, first control point, second control point, and endpoint for each curve.

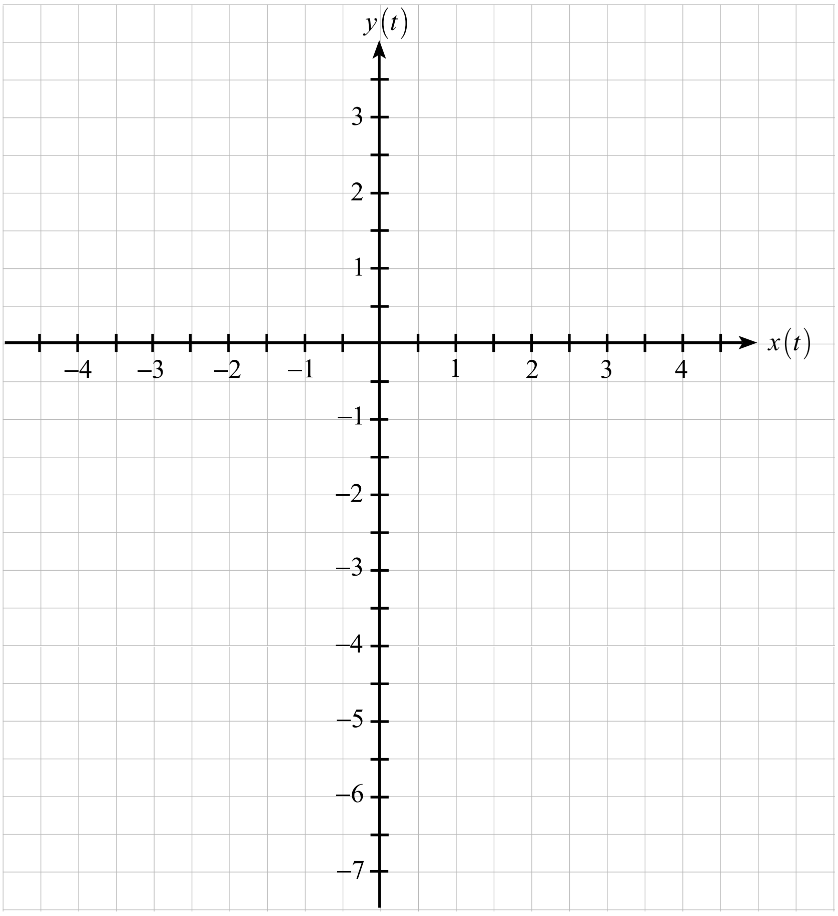


Figure 13 (4 marks)

(d) The designer wants to place another shape inside the curves drawn in part (c). The upper part of the new shape is a curve with parametric equations

$$\begin{cases} x(t) = -2t^3 + 3t^2 + 3t - 2 \\ y(t) = -3t^2 + 3t - 2 \end{cases} \quad \text{where } 0 \le t \le 1.$$

The lower part of the new shape is to be the reflection of this new curve in the line $y = -2$.

(i) Give the parametric equations for the lower part of the new shape.

(Hint: Your answers to parts (a) and (b)(ii) may help.)

(1 mark)

(ii) On the axes in Figure 13, draw the new shape formed by the two new curves to complete this part of the designer's logo.

(1 mark)

(e) For the curve from part (a):

(i) find $\frac{dy}{dx}$ in terms of t.

(2 marks)

(ii) evaluate $\frac{dy}{dx}$ when $t = \frac{1}{2}$.

(1 mark)

(iii) indicate, on Figure 13, the point for which part (e)(ii) applies. (1 mark)

You may write on this page if you need more space to finish your answers. Make sure to label each answer carefully (e.g. 'Question 8(a)(ii) continued').

You may remove this page from the booklet by tearing along the perforations so that you can refer to it while you write your answers.

LIST OF MATHEMATICAL FORMULAE FOR USE IN STAGE 2 SPECIALIST MATHEMATICS

Circular Functions

$\sin^2 A + \cos^2 A = 1$

$\tan^2 A + 1 = \sec^2 A$

$1 + \cot^2 A = \operatorname{cosec}^2 A$

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$\sin 2A = 2 \sin A \cos A$

$$\begin{aligned} \cos 2A &= \cos^2 A - \sin^2 A \\ &= 2\cos^2 A - 1 \\ &= 1 - 2\sin^2 A \end{aligned}$$

$\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$

$2 \sin A \cos B = \sin(A + B) + \sin(A - B)$

$2 \cos A \cos B = \cos(A + B) + \cos(A - B)$

$2 \sin A \sin B = \cos(A - B) - \cos(A + B)$

$\sin A \pm \sin B = 2 \sin \tfrac{1}{2}(A \pm B) \cos \tfrac{1}{2}(A \mp B)$

$\cos A + \cos B = 2 \cos \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B)$

$\cos A - \cos B = -2 \sin \tfrac{1}{2}(A + B) \sin \tfrac{1}{2}(A - B)$

Matrices and Determinants

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\det A = |A| = ad - bc$ and

$A^{-1} = \dfrac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Derivatives

$f(x) = y$	$f'(x) = \dfrac{dy}{dx}$
x^n	nx^{n-1}
e^x	e^x
$\ln x = \log_e x$	$\dfrac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$

Properties of Derivatives

$\dfrac{d}{dx}\{f(x)g(x)\} = f'(x)g(x) + f(x)g'(x)$

$\dfrac{d}{dx}\left\{\dfrac{f(x)}{g(x)}\right\} = \dfrac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

$\dfrac{d}{dx} f(g(x)) = f'(g(x))g'(x)$

Quadratic Equations

If $ax^2 + bx + c = 0$ then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Distance from a Point to a Plane

The distance from (x_1, y_1, z_1) to $Ax + By + Cz + D = 0$ is given by

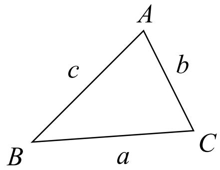
$\dfrac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$.

Mensuration

Area of sector $= \tfrac{1}{2} r^2 \theta$

Arc length $= r\theta$
(where θ is in radians)

In any triangle ABC:



Area of triangle $= \tfrac{1}{2} ab \sin C$

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

$a^2 = b^2 + c^2 - 2bc \cos A$