SSABSA
SENIOR SECONDARY ASSESSMENT BOARD OF SOUTH AUSTRALIA
EXTERNAL EXAMINATION 2007

2007 SPECIALIST MATHEMATICS

FOR OFFICE USE ONLY

SUPERVISOR CHECK

RE-MARKED

ATTACH SACE REGISTRATION NUMBER LABEL TO THIS BOX

Graphics calculator
Brand
Model
Computer software

Friday 16 November: 9 a.m.

Time: 3 hours

Examination material: one 41-page question booklet
one SACE registration number label

Pages: 41
Questions: 16

Approved dictionaries, notes, calculators, and computer software may be used.

Instructions to Students

1. You will have 10 minutes to read the paper. You must not write in your question booklet or use a calculator during this reading time but you may make notes on the scribbling paper provided.

2. This paper consists of three sections:

   Section A (Questions 1 to 10) 75 marks
   Answer ***all*** questions in Section A.

   Section B (Questions 11 to 14) 60 marks
   Answer ***all*** questions in Section B.

   Section C (Questions 15 and 16) 15 marks
   Answer ***one*** question from Section C.

3. Write your answers in the spaces provided in this question booklet. There is no need to fill all the space provided. You may write on page 40 if you need more space, making sure to label each answer clearly.

4. Appropriate steps of logic and correct answers are required for full marks.

5. Show all working in this booklet. (You are strongly advised ***not*** to use scribbling paper. Work that you consider incorrect should be crossed out with a single line.)

6. Use only black or blue pens for all work other than graphs and diagrams, for which you may use a sharp dark pencil.

7. State all answers correct to three significant figures, unless otherwise stated or as appropriate.

8. Diagrams, where given, are not necessarily drawn to scale.

9. The list of mathematical formulae is on page 41. You may remove the page from this booklet before the examination begins.

10. Complete the box on the top right-hand side of this page with information about the electronic technology you are using in this examination.

11. Attach your SACE registration number label to the box at the top of this page.

SECTION A (Questions 1 to 10)
(75 marks)

*Answer **all** questions in this section.*

QUESTION 1 (5 marks)

(a) Find in parametric form the equation of the line through $A(2, 6, -2)$ and $B(5, 0, 7)$.

(2 marks)

(b) Find where the line in part (a) intersects the plane $x + 3y + 2z = 14$.

(3 marks)

QUESTION 2 (5 marks)

Consider the function $f(x) = e^{x^2}\sin^3 x$.

(a) Find an approximate value for $\int_{\frac{\pi}{2}}^{\pi} e^{x^2}\sin^3 x\,\mathrm{d}x$.

(1 mark)

(b) Show that $f(x)$ is an odd function.

(2 marks)

(c) Hence find an exact value of k such that

$$\int_{-\pi}^{k} f(x)\mathrm{d}x = -\int_{\frac{\pi}{2}}^{\pi} f(x)\mathrm{d}x,$$ explaining your reasoning.

(2 marks)

QUESTION 3 (6 marks)

In Figure 1 points *A*, *B*, *C,* and *D* lie on the circumference of a circle and *AB* is parallel to *DC*.

The line *AT* is a tangent to the circle and *AD* bisects $\angle TAC$.

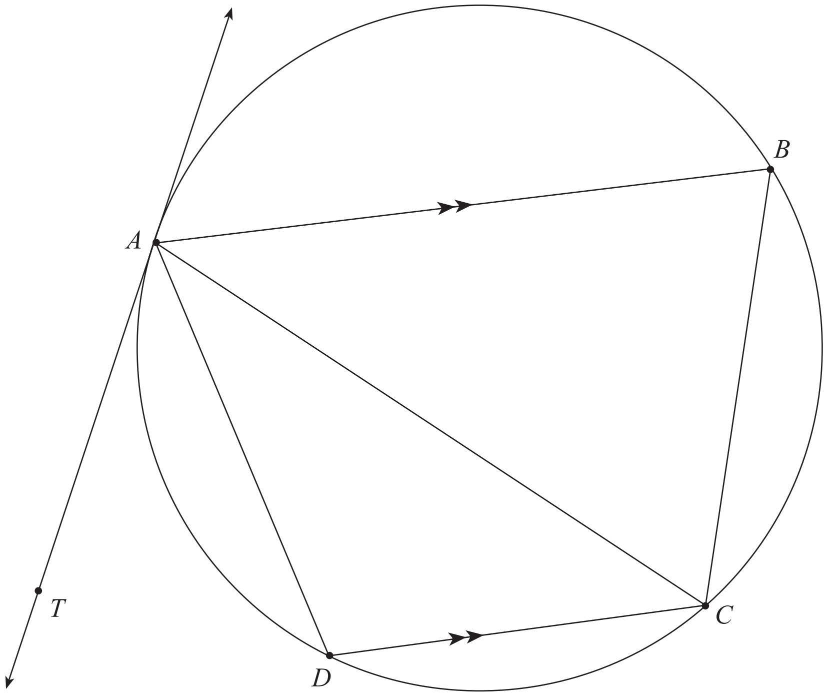

Figure 1

(a) Prove that *AC* bisects $\angle DAB$.

(3 marks)

(b) Prove that $AD = DC = BC$.

(3 marks)

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QUESTION 4 (6 marks)

(a) Prove that if the line $\frac{x-k}{a} = \frac{y-l}{b} = \frac{z-m}{c}$ is parallel to the plane $Ax + By + Cz = D$, then $Aa + Bb + Cc = 0$.

(2 marks)

(b) For the line $\frac{x-3}{3} = \frac{y+4}{4} = \frac{z-m}{c}$ and the plane $2x - y + z = 18$:

(i) find c if the line is parallel to the plane.

(2 marks)

(ii) find m if the line is in the plane.

(2 marks)

QUESTION 5 (5 marks)

(a) If $y = \ln(\sin x)$, where $0 < x < \pi$, show that $\frac{dy}{dx} = \cot x$.

(2 marks)

(b) Hence, or otherwise, find an exact value for $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cot x \, dx$.

(3 marks)

QUESTION 6 (9 marks)

(a) Given that $w = \frac{\sqrt{3}}{2} + \frac{3i}{2}$, write w in the form $r\,\text{cis}\,\theta$, with exact values for r and θ.

(2 marks)

(b) Given the set of complex numbers z such that $|z - 2i| = 1$:

(i) sketch $|z - 2i| = 1$ on the Argand diagram in Figure 2.

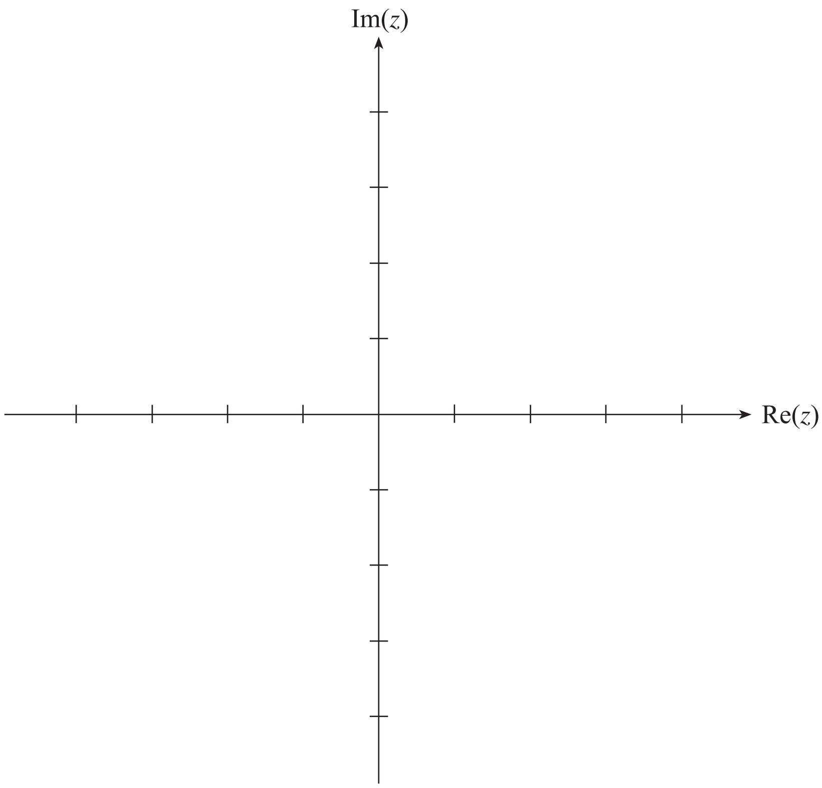

Figure 2

(3 marks)

(ii) show that $\arg z = \arg w$ is a tangent to $|z - 2i| = 1$.

(2 marks)

(iii) find the largest possible value of $\arg z$. Give reasons for your answer.

(2 marks)

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QUESTION 7 (9 marks)

A damaged oil tanker is leaking oil into the sea. A constant current pushes the spreading oil spill into the shape of a sector with a radius of r metres and a sector angle of θ radians (as shown in Figure 3), where r and θ change with time.

Let the area of the sector be A.

Source: International MARSAC, www.marsac.nl/REF.htm

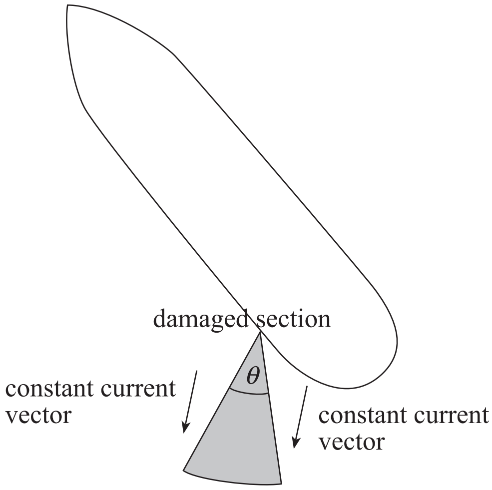

Figure 3

(a) Show that $\dfrac{d\theta}{dt} = \dfrac{2}{r}\left(\dfrac{1}{r}\dfrac{dA}{dt} - \theta\dfrac{dr}{dt}\right)$.

(3 marks)

(b) The radius of the oil spill is increasing at a constant rate of 2 metres per second and the area of the oil spill is increasing at a constant rate of 2π square metres per second.

Consider the oil spill when it has a radius of 6 metres.

(i) Show that the area of the oil spill is $A = 6\pi$ square metres at this instant.

(2 marks)

(ii) Hence find the exact value of the sector angle, θ, at this instant.

(2 marks)

(iii) Hence find the rate of change of θ at this instant.

(2 marks)

QUESTION 8 (10 marks)

(a) Let $f(x) = x^3 + 7$.

Show that $f(x) - f(2)$ has a factor of $x - 2$.

(2 marks)

(b) If $p(x)$ is any polynomial of degree ≥ 1, prove that $p(x) - p(k)$ has a factor of $x - k$.

(2 marks)

(c) $T(x)$ is a real cubic polynomial with a zero of $1 + 2i$.

(i) Find a real quadratic factor of $T(x)$.

(3 marks)

(ii) Find $T(x)$, given that $T(x)-25$ has a factor of $x-2$ and that $T(x)-12$ has a factor of $x-1$.

(3 marks)

QUESTION 9 (10 marks)

Points $A(-1, 1)$, $B(1, 2)$, and $C(2, -1)$ are fixed points in the plane which determine the simultaneous motion of points P, Q, and R so that

$$\overrightarrow{OP} = [2t-1,\ t+1]$$

$$\overrightarrow{OQ} = [t+1,\ -3t+2]$$

$$\overrightarrow{OR} = [-t^2+4t-1,\ -4t^2+2t+1]$$

where $0 \le t \le 1$ is the time for which the points are in motion.

The graph in Figure 4 represents this situation at some time t.

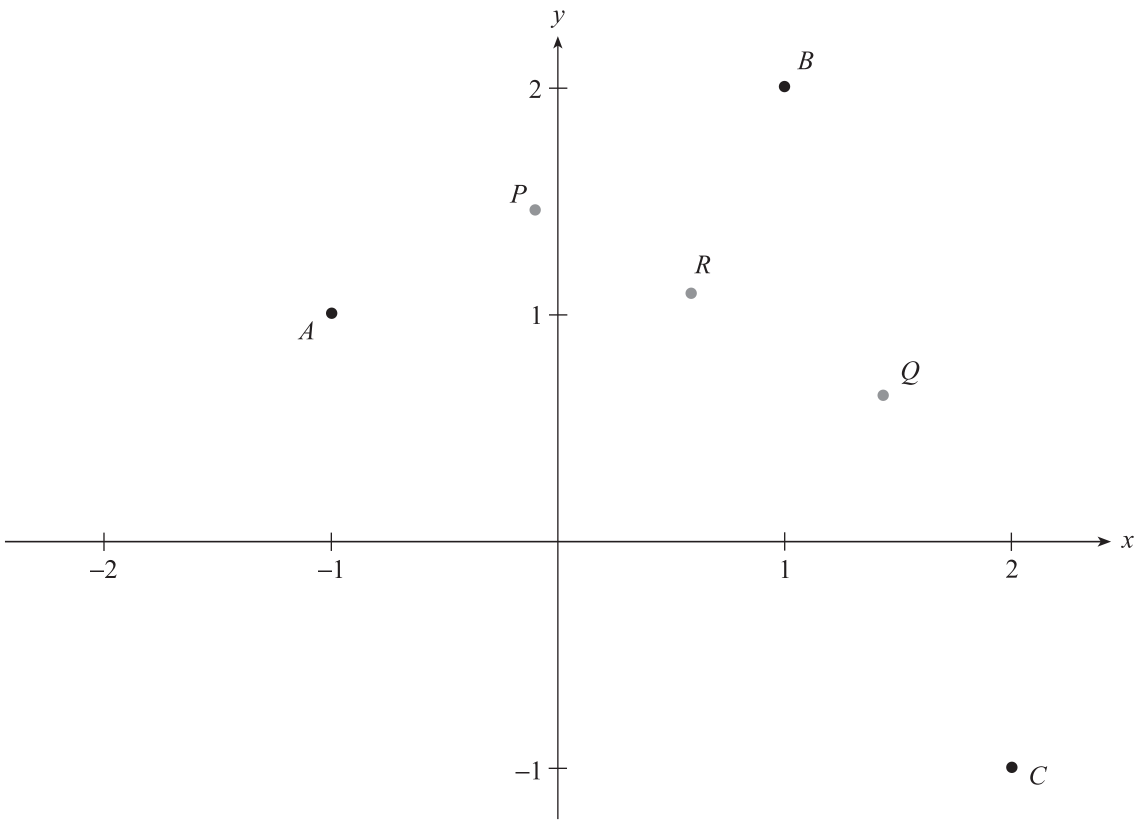

Figure 4

(a) Graph the paths of P, Q, and R on the axes in Figure 4. (3 marks)

(b) Find $\overrightarrow{PR}$ and $\overrightarrow{PQ}$ in terms of t and hence give a vector proof that P, R, and Q are collinear.

(3 marks)

(c) (i) Draw the vectors $\overrightarrow{AP}$, $\overrightarrow{AR}$, $\overrightarrow{PQ}$, $\overrightarrow{RC}$, and $\overrightarrow{QC}$ on Figure 4. (1 mark)

(ii) Using the triangle inequality, show that $|\overrightarrow{AP}|+|\overrightarrow{PQ}|+|\overrightarrow{QC}| \geq |\overrightarrow{AR}|+|\overrightarrow{RC}|$.

(3 marks)

QUESTION 10 (10 marks)

The velocity, v, of a small raindrop may be found by solving the differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} = g - cv$$

where g is the gravitational constant, c is a positive constant that can be found experimentally, and time, t, is measured in seconds.

Source: Photograph by Altrendo Nature, from Getty Images, http://creative.gettyimages.com

(a) By solving the differential equation with the initial condition of $v(0) = 0$, show that the velocity of a small raindrop is

$$v(t) = \frac{g}{c}\left(1 - e^{-ct}\right) \text{ units per second.}$$

(5 marks)

(b) The limiting value of the velocity as time increases is called the terminal velocity.

Find the terminal velocity of a small raindrop using the solution from part (a), given that $g = 9.8$ and $c = 16$.

(1 mark)

(c) On the axes in Figure 5, sketch the velocity curve of a small raindrop using the values of g and c given in part (b).

Indicate the terminal velocity on the sketch.

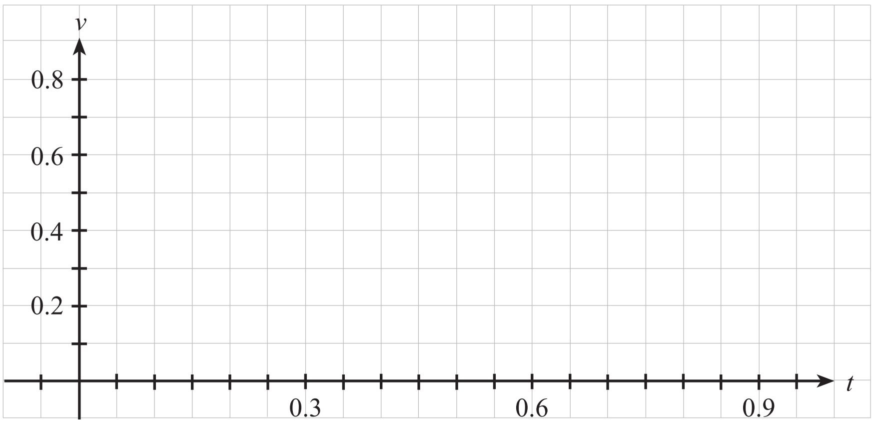

Figure 5

(3 marks)

(d) Small raindrops approach terminal velocity rapidly.

By assuming that the velocity of a small raindrop is constant, find an approximate value for the time taken for a small raindrop to fall a distance of 10 units.

(1 mark)

SECTION B (Questions 11 to 14)
(60 marks)

*Answer **all** questions in this section.*

QUESTION 11 (15 marks)

Figure 6 shows skew lines l_1 and l_2 connected by vector $\overrightarrow{NM}$.

Line l_1 is parallel to $\boldsymbol{v} = [1, 1, 4]$ and has vector equation

$$l_1 : [x, y, z] = [1, 6, -9] + s[1, 1, 4].$$

Line l_2 is parallel to $\boldsymbol{w} = [2, -1, 2]$ and has parametric equations

$$l_2 : \begin{cases} x = 2 + 2t \\ y = 1 - t \\ z = 1 + 2t. \end{cases}$$

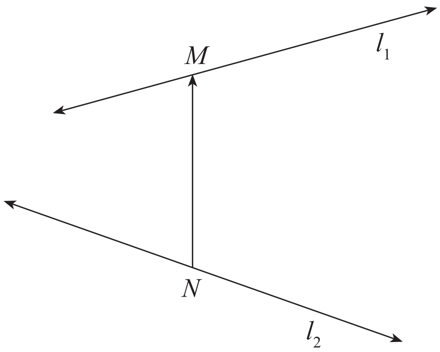

Figure 6

(a) (i) Calculate $\boldsymbol{v} \times \boldsymbol{w}$.

(2 marks)

(ii) Show that point $P(4, 9, 3)$ lies on l_1 and point $Q(10, -3, 9)$ lies on l_2.

(2 marks)

(iii) Calculate the length of the projection of vector $\overrightarrow{PQ}$ on $\boldsymbol{v} \times \boldsymbol{w}$.

(2 marks)

(b) As shown in Figure 6, M is a point on l_1 and N is a point on l_2.
Show that $\overrightarrow{NM} = [s - 2t - 1,\ s + t + 5,\ 4s - 2t - 10]$.

(1 mark)

(c) (i) If $\overrightarrow{NM} = k[2,\ 2, -1]$, show that s, t, and k are related by the system of equations

$$s - 2t - 2k = 1$$
$$s + t - 2k = -5$$
$$4s - 2t + k = 10.$$

(1 mark)

PLEASE TURN OVER

(ii) Solve the system of equations from part (c)(i) for s, t, and k.

(1 mark)

(iii) Hence find the coordinates of M and N.

(2 marks)

(iv) Find $\overrightarrow{NM}$ and calculate the length of this vector.

(2 marks)

(d) (i) Comment on your answers to part (a)(iii) and part (c)(iv).

(1 mark)

(ii) Explain why $\left|\overrightarrow{NM}\right|$ found in part (c)(iv) is the shortest distance between l_1 and l_2.

(1 mark)

QUESTION 12 (16 marks)

(a) (i) Solve $z^6 = -64$, giving the roots in the form $r \operatorname{cis} \theta$.

Illustrate the roots on the Argand diagram in Figure 7.

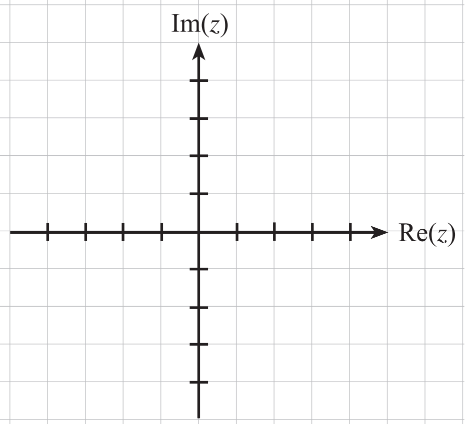

Figure 7

(5 marks)

(ii) Show that $z^2 + 4$ is a factor of $z^6 + 64$.

(2 marks)

(iii) Using parts (a)(i) and (ii), or otherwise, solve $z^4 - 4z^2 + 16 = 0$, giving your answers in the form $r\operatorname{cis}\theta$.

(2 marks)

(b) (i) Show that if $p(z) = (z - z_1)(z - z_2)$ is a quadratic polynomial with zeros z_1 and z_2, then the coefficient of z in $p(z)$ is $-(z_1 + z_2)$.

(1 mark)

(ii) Let $p(z) = (z - z_3)(z^2 + az + b)$ be a cubic polynomial, where a and b are constants. If $p(z)$ has zeros z_1, z_2, and z_3 then, using part (b)(i), show that the coefficient of z^2 in $p(z)$ is $-(z_1 + z_2 + z_3)$.

(1 mark)

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(iii) Use an inductive argument to show that if $p(z) = z^n + az^{n-1} + bz^{n-2} + \ldots$ is a polynomial of degree n with zeros $z_1, z_2, z_3, z_4, \ldots, z_n$, then the coefficient of z^{n-1} in $p(z)$ is $-(z_1 + z_2 + z_3 + \ldots + z_n)$.

(3 marks)

(c) If z_1, z_2, z_3, z_4, z_5, and z_6 are the roots of $z^6 = -64$ from part (a)(i), find:

(i) $z_1 + z_2 + z_3 + z_4 + z_5 + z_6$.

(1 mark)

(ii) $z_1 z_2 z_3 z_4 z_5 z_6$.

(1 mark)

QUESTION 13 (14 marks)

(a) A mathematical model for the growth of a population $P = P(t)$ of fruit flies in a laboratory is given by

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{49}{500}P\left(\frac{1000 - P}{1000}\right)$$

where time is measured in days and the maximum number of fruit flies able to be sustained in the laboratory is 1000.

Initially there are 100 fruit flies present. That is $P(0) = 100$.

(i) Figure 8 shows the slope field for the differential equation given above.

Draw the solution curve on the slope field.

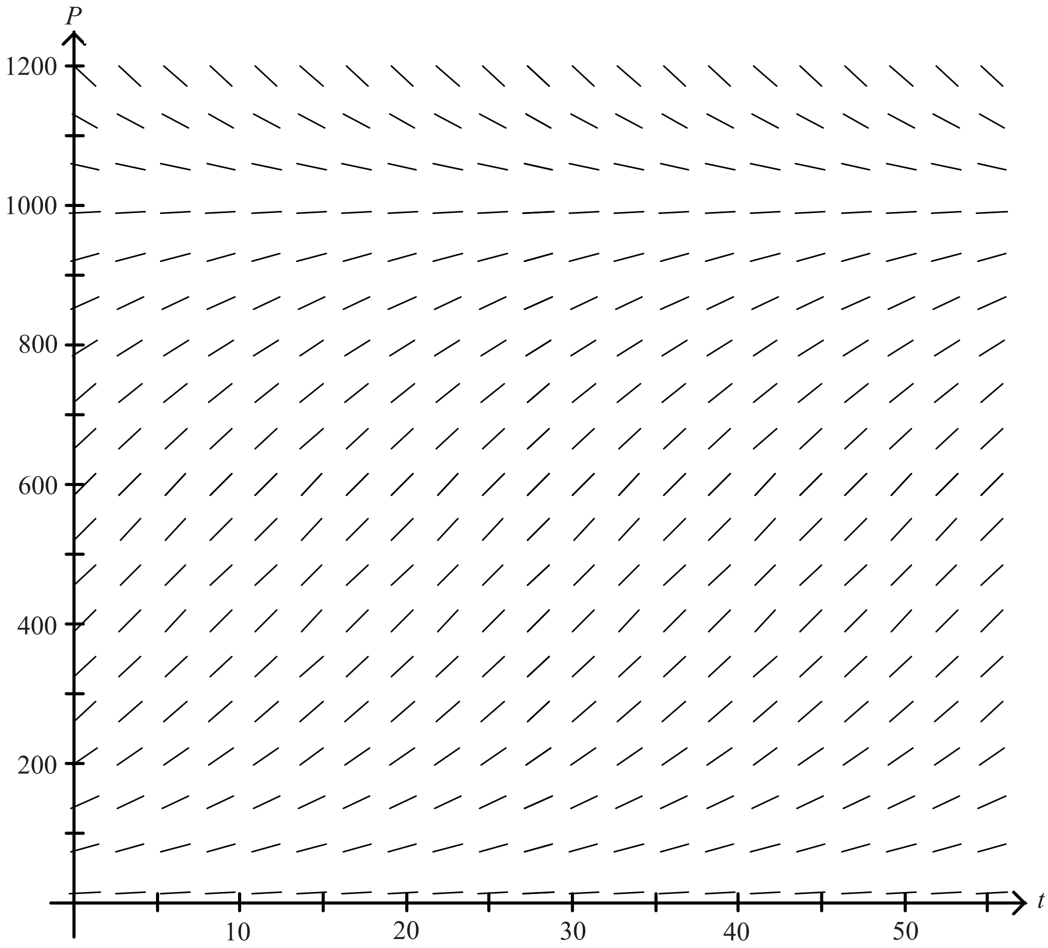

Figure 8 (3 marks)

(ii) Show that $\dfrac{1}{P} + \dfrac{1}{1000 - P} = \dfrac{1000}{P(1000 - P)}$.

(1 mark)

(iii) Solve the differential equation $\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{49}{500} P\left(\frac{1000 - P}{1000}\right)$ to show that

$$P(t) = \frac{1000}{1 + 9e^{-0.098t}}.$$

(5 marks)

(iv) How many days does it take for the population of fruit flies to reach 500?

(1 mark)

(b) More generally, consider a model for the growth of a population of fruit flies to be

$$\frac{dP}{dt} = \frac{kP(1000 - P)}{1000}, \text{ where } P(0) = 100 \text{ and } k > 0 \text{ is a constant.}$$

Let T represent the time taken for the population to reach 500 fruit flies.

The approximate relationship between T and k is $T \approx \frac{2.20}{k}$ days.

(i) To what level of accuracy does this relationship hold for your answer to part (a)(iv)?

(1 mark)

(ii) Suggest a value of k for which T will be less than your answer to part (a)(iv).

(1 mark)

(c) After thirty days 600 fruit flies are removed from the population described in part (a)(iii).

Using the value of k from part (b)(ii), state a new differential equation with a new initial condition that could be used to model the growth of the remaining population.

(2 marks)

QUESTION 14 (15 marks)

While designing an animated advertisement for Fable 8 Fantasy Bookstores, a computer graphics specialist uses a moving point controlled by the parametric equations

$$x = \sin 2t,\ y = \cos t$$

where $0 \le t \le 2\pi$ is the time taken for the point to complete one circuit of the curve shown in Figure 9.

The curve $x = \sin 2t$, $y = \cos t$

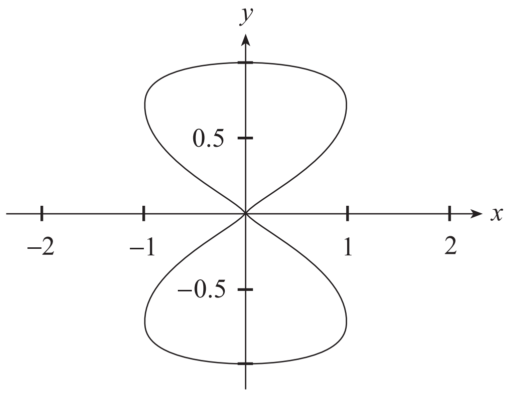

Figure 9

(a) Find the velocity vector for the moving point.

(2 marks)

(b) (i) Hence show that $s(t)$, the speed of the moving point at time t, is given by

$$s(t) = \sqrt{16\sin^4 t - 15\sin^2 t + 4}.$$

(3 marks)

(ii) Graph $s(t)$ on the axes in Figure 10.

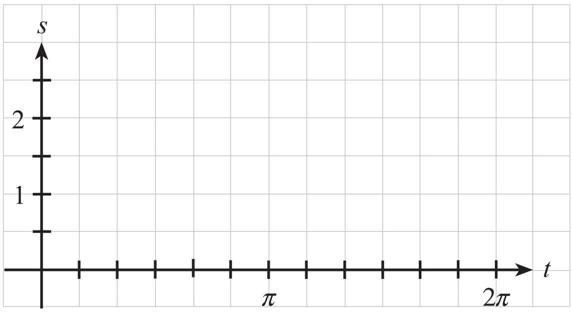

Figure 10

(3 marks)

(c) (i) Show that $s'(t) = \dfrac{\sin 2t\left(32\sin^2 t - 15\right)}{2s(t)}$.

(4 marks)

PLEASE TURN OVER

(ii) Hence find exact values for the maximum speed and minimum speed of the moving point as it completes one circuit of the curve shown in Figure 9.

(3 marks)

SECTION C (Questions 15 and 16)
(15 marks)

Answer ***one*** *question from this section,* ***either*** *Question 15* ***or*** *Question 16.*

Answer ***either*** *Question 15* ***or*** *Question 16.*

QUESTION 15 (15 marks)

Figure 11 shows the slope field for the differential system $\begin{array}{l} x' = x + y \\ y' = -2x - y \end{array}$ where $0 \le t \le 2\pi$.

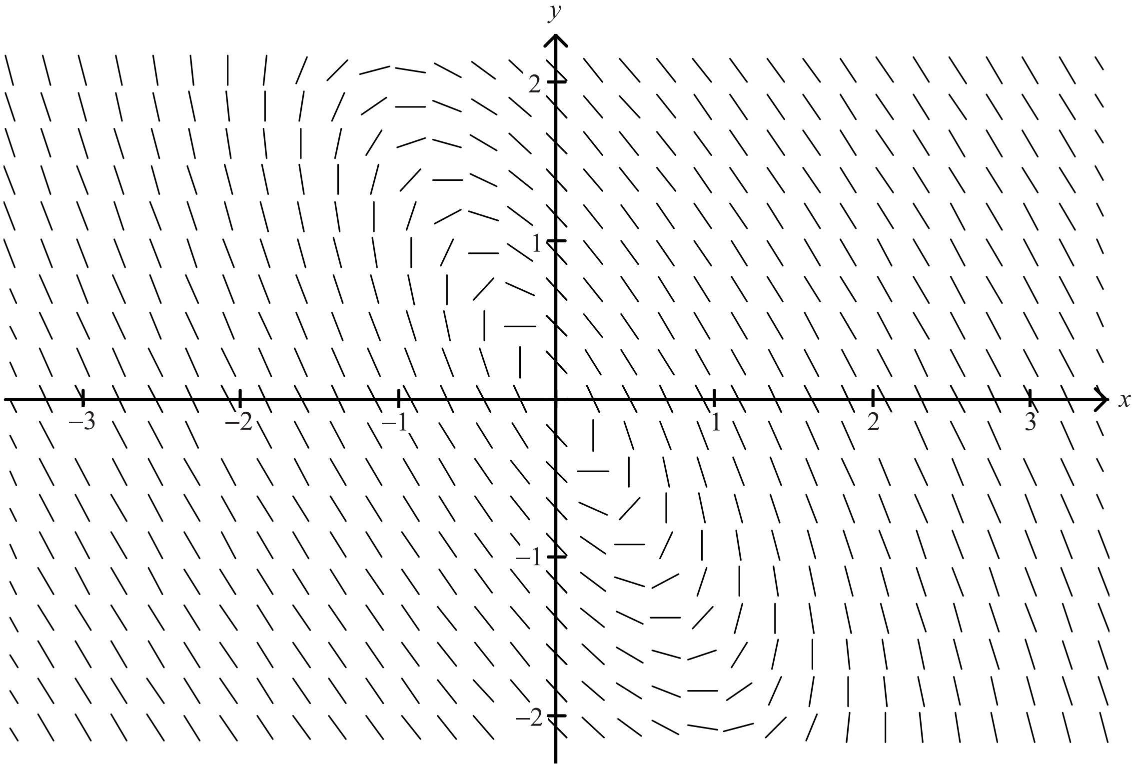

Figure 11

(a) On Figure 11 draw the solution curve that passes through the point (1, 0). (3 marks)

(b) Show that $\dfrac{dy}{dx} = \dfrac{-2x - y}{x + y}$.

(1 mark)

For a function $y = f(x)$ obeying the differential equation $\frac{dy}{dx} = f'(x)$, the equations for Euler's method are

$$x_n = x_{n-1} + h$$
$$y_n = y_{n-1} + hf'(x_{n-1})$$

where h is a number of sufficiently small size.

In the current situation the equations for Euler's method can be adapted to

$$x_n = x_{n-1} + h$$
$$y_n = y_{n-1} + h\left(\frac{-2x_{n-1} - y_{n-1}}{x_{n-1} + y_{n-1}}\right).$$

With $h = -0.1$, these equations can be used to find an estimate for the positive y-intercept of the solution curve that you drew in part (a).

(c) Complete the last column of the table below to find an estimate for the positive y-intercept of the solution curve. Only the results for the first three calculations and the last three calculations are needed. You do not need to fill in any of the shaded cells, but you may use them if you wish.

n	x_{n-1}	y_{n-1}	h	$\frac{-2x_{n-1} - y_{n-1}}{x_{n-1} + y_{n-1}}$	y_n
1	1.0	0	–0.1	–2.0000	0.2000
2	0.9	0.2000	–0.1		
3					
8					
9					
10					

(4 marks)

(d) It can be shown that the given differential system

$$\begin{aligned} x' &= x + y \\ y' &= -2x - y \end{aligned} \quad \text{where } 0 \le t \le 2\pi$$

has a solution of the form $\begin{cases} x(t) = A\cos t + B\sin t \\ y(t) = C\cos t + D\sin t \end{cases}$ where A, B, C, and D are constants.

(*Note*: You do not have to prove this.)

The solution curve that you drew in part (a) has initial conditions $x(0) = 1, y(0) = 0$.

(i) Find the values for A, B, C, and D.

(3 marks)

(ii) Hence find the exact value of the positive y-intercept of the solution curve.

(2 marks)

(iii) Find the Cartesian equation for the solution curve.

(2 marks)

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Answer ***either*** *Question 15* ***or*** *Question 16.*

QUESTION 16 (15 marks)

(a) Find $(2-i)^2$.

(1 mark)

(b) Consider the complex iteration $z \to \frac{1}{2}\left(z + \frac{c}{z}\right)$.

(i) Complete the table of iterates below with $z_0 = 1$ and $c = 3 - 4i$.

n	z_n
0	1
1	
2	
3	
4	
5	
6	
7	

(2 marks)

(ii) Complete the table of iterates below with $z_0 = i$ and $c = 3 - 4i$.

n	z_n
0	i
1	
2	
3	
4	
5	
6	
7	

(2 marks)

(c) (i) Show that, in general, the complex iteration $z \to \frac{1}{2}\left(z + \frac{c}{z}\right)$ has invariant points $z = \pm\sqrt{c}$.

(2 marks)

(ii) Hence explain the results obtained for part (a) and parts (b)(i) and (ii).

(1 mark)

(d) (i) State the invariant points for the iteration $z \to \frac{1}{2}\left(z + \frac{4}{z}\right)$.

(1 mark)

(ii) Complete the table of iterates for the iteration $z \to \frac{1}{2}\left(z + \frac{4}{z}\right)$ with $z_0 = 1$.

n	z_n
0	1
1	
2	
3	
4	
5	
6	
7	

(2 marks)

(iii) For the iteration $z \to \frac{1}{2}\left(z + \frac{4}{z}\right)$, let $z_n = 2 + a$, where a is a non-zero number of small size.

Show that $z_{n+1} = 2 + \frac{a^2}{4 + 2a}$.

(2 marks)

(iv) Hence explain that if $z_n \neq 2$, then $z_{n+1} \neq 2$.

(1 mark)

(v) Explain any apparent contradiction in your answers to parts (d)(i), (ii), and (iv).

(1 mark)

You may write on this page if you need more space to finish your answers. Make sure to label each answer carefully (e.g. 'Question 8(c)(ii) continued').

You may remove this page from the booklet by tearing along the perforations.

LIST OF MATHEMATICAL FORMULAE FOR USE IN STAGE 2 SPECIALIST MATHEMATICS

Circular Functions

$\sin^2 A + \cos^2 A = 1$

$\tan^2 A + 1 = \sec^2 A$

$1 + \cot^2 A = \operatorname{cosec}^2 A$

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$\sin 2A = 2 \sin A \cos A$

$\cos 2A = \cos^2 A - \sin^2 A$

$\quad = 2\cos^2 A - 1$

$\quad = 1 - 2\sin^2 A$

$\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$

$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$

$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$

$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$

$\sin A \pm \sin B = 2 \sin \tfrac{1}{2}(A \pm B) \cos \tfrac{1}{2}(A \mp B)$

$\cos A + \cos B = 2 \cos \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}(A-B)$

$\cos A - \cos B = -2 \sin \tfrac{1}{2}(A+B) \sin \tfrac{1}{2}(A-B)$

Matrices and Determinants

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\det A = |A| = ad - bc$ and

$A^{-1} = \dfrac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Derivatives

$f(x) = y$	$f'(x) = \dfrac{dy}{dx}$
x^n	nx^{n-1}
e^x	e^x
$\ln x = \log_e x$	$\dfrac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$

Properties of Derivatives

$\dfrac{d}{dx}\{f(x)\, g(x)\} = f'(x)\, g(x) + f(x)\, g'(x)$

$\dfrac{d}{dx}\left\{\dfrac{f(x)}{g(x)}\right\} = \dfrac{f'(x)\, g(x) - f(x)\, g'(x)}{[g(x)]^2}$

$\dfrac{d}{dx} f(g(x)) = f'(g(x))\, g'(x)$

Quadratic Equations

If $ax^2 + bx + c = 0$ then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Distance from a Point to a Plane

The distance from (x_1, y_1, z_1) to $Ax + By + Cz + D = 0$ is given by

$\dfrac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$.

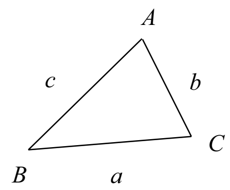
Mensuration

Area of sector $= \tfrac{1}{2} r^2 \theta$

Arc length $= r\theta$

(where θ is in radians)

In any triangle ABC:



Area of triangle $= \tfrac{1}{2} ab \sin C$

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

$a^2 = b^2 + c^2 - 2bc \cos A$